UNIT - III

Couples - Resultant of several Coplanar forces - Equation of the line of action of the resultant - equilibrium of a rigid body under three coplanar forces.

(A) Definition: Couple:

Two equal and unlike parallel forces not acting at the same point are said to form a couple.

Examples of a couple:

Forces used in winding a clock or turning a tap are examples of a couple.

Such forces acting upon a rigid body can have only a rotatory effect on the body. They cannot produce a motion of translation.

Remarks:

1. Since a couple cannot be reduced to a single force, a couple is not equivalent to any single force. So a couple and a force will never keep a rigid body in equilibrium.
2. The vector sum of the constituent forces of a couple is zero.

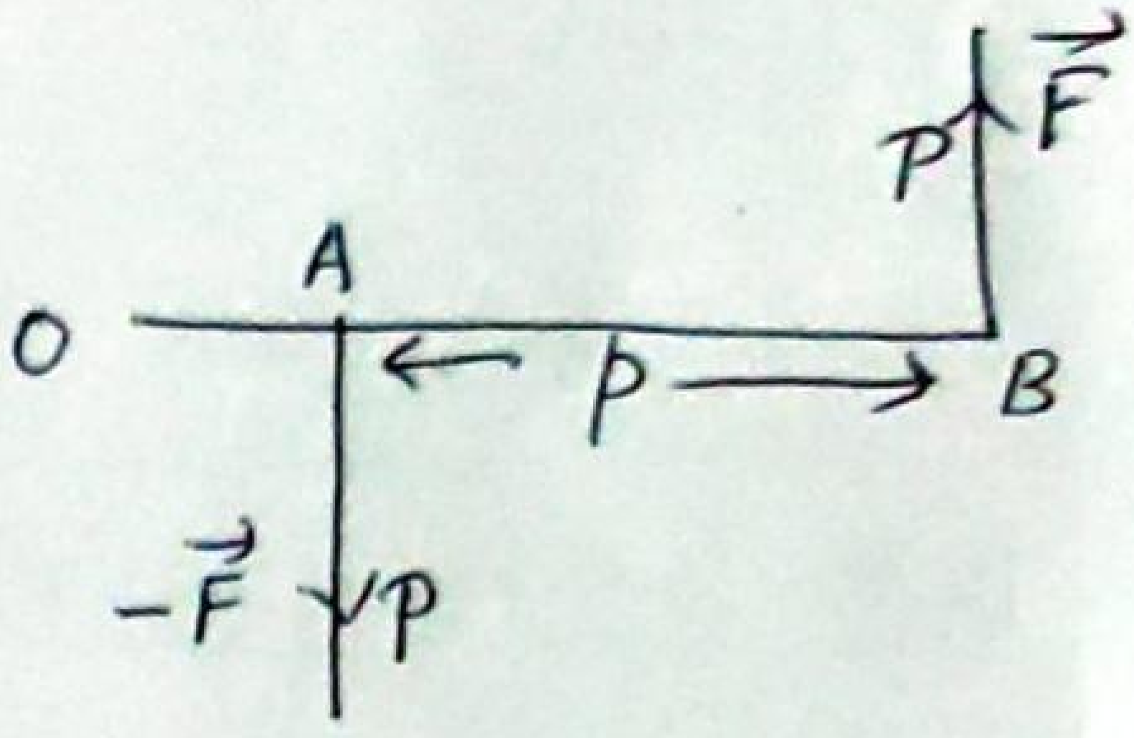
Moment of a Couple:

Definition: The sum of the moments of the constituent forces of a couple about any point is called the moment of the couple.

Bookwork: 1

To show that the moment of a couple is independent of the point about which the moment is obtained.

Proof:



Consider a couple $(-\vec{F}, \vec{F})$ constituted by the forces $-\vec{F}$ and $\vec{F}$.

Let P be the magnitudes of the two equal forces. Let O be any point in their plane.

Draw OAB perpendicular to the forces to meet the lines of action in A and B.

The algebraic sum of the moments of the forces about O $= P \cdot OB - P\,OA$.

$$= P(OB - OA)$$
$$= P \cdot AB$$
$$= Pp.$$

This value is independent of the position of O.

Alternate definition of a moment of a couple:

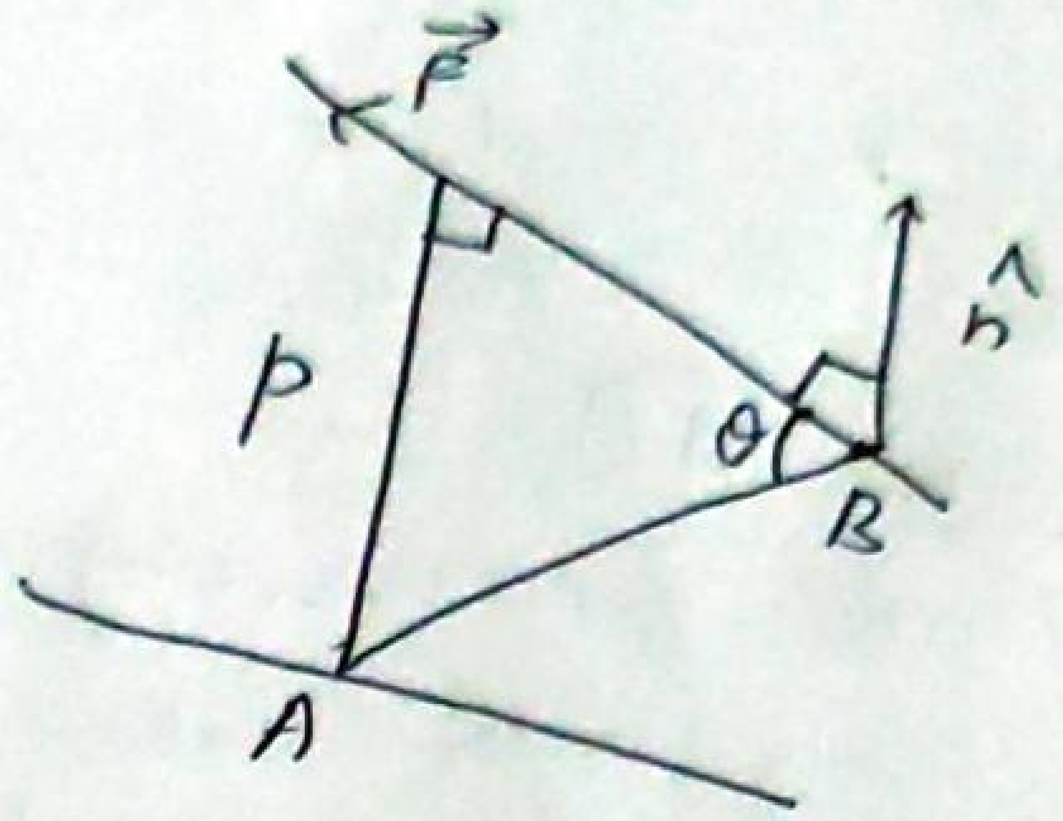
The moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them.

Remark: $\left\{ \begin{array}{l} \text{Moment of} \\ \text{a couple} \end{array} \right\} = \left\{ \begin{array}{l} \text{Moment of one constituent} \\ \text{force about a point on the} \\ \text{line of action of the other} \end{array} \right\}$.

Arm and axis of a Couple:

If $\hat{n}$ is the unit vector perpendicular to the plane of a couple such that AB, $\vec{F}$, $\hat{n}$ form a right-handed triad, then the moment of the Couple is

$$\vec{AB} \times \vec{F} = AB \cdot F \cdot \sin\theta \cdot \hat{n}$$
$$= (AB \sin\theta) F \cdot \hat{n}$$
$$= p \cdot F \cdot \hat{n}$$



where θ is the angle between $\vec{AB}$ and $\vec{F}$.

$$\sin\theta = \frac{p}{AB}.$$
$$\Rightarrow p = AB \sin\theta.$$

p is the perpendicular distance between the lines of action of the forces.

p is called the arm of the Couple, the direction of $\hat{n}$ is called the axis of the couple.

Definition: Arm of the couple.

The perpendicular distance between the lines of action of the forces forming a couple is called the arm of the Couple, denoted by p.

Definition: Axis of the couple:

The direction of the unit vector perpendicular to the plane of a couple is called the axis of the Couple, denoted by $\hat{n}$.

Remark: A Couple each of whose forces is P and whose arm is p is denoted by (P, p).

Remark:

A couple is positive when its moment is positive and is negative when its moment is negative.

(ie) If the forces of the couple tend to produce rotation in the anticlockwise direction, the couple is positive and a couple is negative when the forces tend to produce rotation in the clockwise direction.

Problem: A1

Three forces acting along the sides of a triangle in the same order are equivalent to a couple. Show that they are proportional to the sides of the triangle.

Solution: Let the triangle be ABC and the forces be $P\hat{BC}$, $Q\hat{CA}$, $R\hat{AB}$.

Since they form a couple, their sum is equal to zero.

So $P\hat{BC} + Q\hat{CA} + R\hat{AB} = 0. \rightarrow (1)$

But, we know that $\overline{AB} + \overline{BC} + \overline{CA} = 0$

$\therefore \quad a\hat{BC} + b\hat{CA} + c.\hat{AB} = 0 \rightarrow (2).$

Comparing the coefficients of $\hat{BC}$, $\hat{CA}$, $\hat{AB}$ in (1) and (2), we get

$$\boxed{\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}}$$

(B) Resultant of several Coplanar forces.

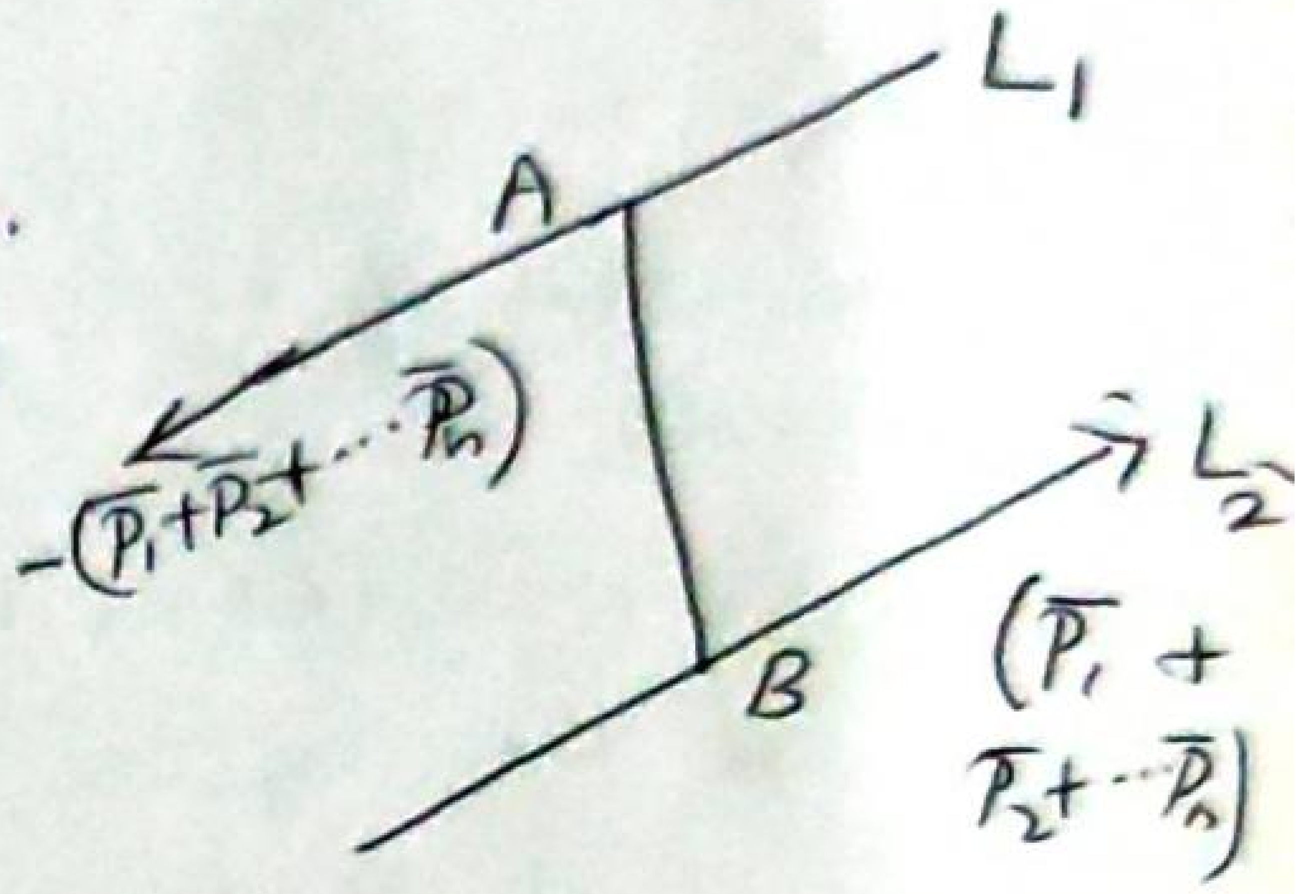
Bookwork: 2

A system of coplanar forces reduce either to a single force or to a Couple.

Proof:

Let L_1 and L_2 be two arbitrary chosen parallel lines in the plane of the Couples.

Let A and B be two points on them.



Replace the given couples by equivalent Couples $(\bar{P}_1, -\bar{P}_1)$, $(P_2, -\bar{P}_2)$, ... $(\bar{P}_n, -\bar{P}_n)$ respectively having the Constituent forces $(-\bar{P}_1, -\bar{P}_2, \cdots -\bar{P}_n)$ along L_1 and $\bar{P}_1, \bar{P}_2, \cdots \bar{P}_n$ along L_2.

We know that the moments of equivalent Couples are equal.

So $\quad \bar{AB} \times \bar{P}_1 = \bar{G}_1, \quad \bar{AB} \times \bar{P}_2 = \bar{G}_2, \quad \cdots \quad \bar{AB} \times \bar{P}_n = \bar{G}_n$.

The resultant of the n forces along L_1 is $-(\bar{P}_1 + \bar{P}_2 + \cdots \bar{P}_n)$ and the resultant of n forces along L_2 is $(\bar{P}_1 + \bar{P}_2 + \cdots + \bar{P}_n)$.

They form a Couple which is equivalent to the given n couples whose moment is

$$\bar{AB} \times (\bar{P}_1 + \bar{P}_2 + \cdots + \bar{P}_n)$$

$$= \bar{AB} \times \bar{P}_1 + \bar{AB} \times \bar{P}_2 + \cdots + \bar{AB} \times \bar{P}_n$$

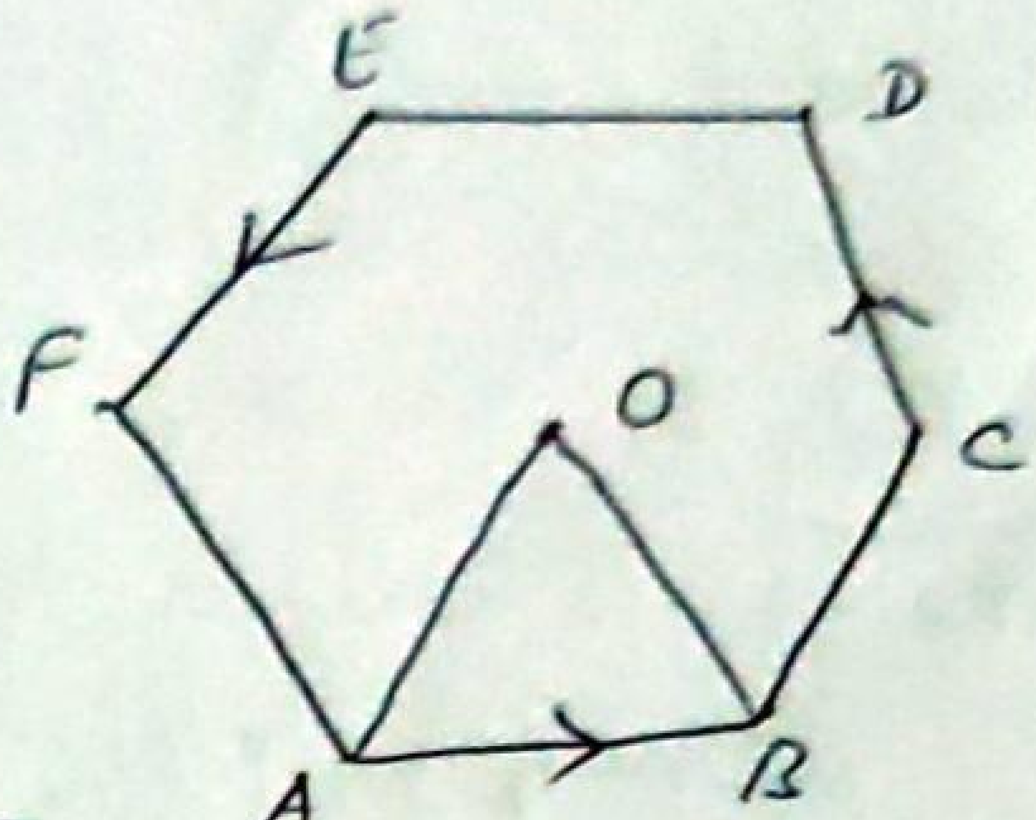
$$= \bar{G}_1 + \bar{G}_2 + \cdots + \bar{G}_n$$

= the sum of the moments of the given couples.

Problems: (B1)

Show that the forces $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ acting respectively at A, C, E of a regular hexagon ABCDEF, are equivalent to a couple of moment equal to the area of the hexagon.

Solution:



Let O be the centre of the hexagon.

The sum of the given forces is

$\overline{AB} + \overline{CD} + \overline{EF} = \overline{AB} + \overline{BO} + \overline{OA} = 0.$

$\Rightarrow$ The system is in equilibrium (or) it reduces to a couple.

But the moment about O is

$= \overline{OA} \times \overline{AB}$

$= OA \cdot AB \cdot \sin OAB\, \hat{k}$ $\left[\because \text{Area of a triangle} = \frac{1}{2} bc \sin A\right]$

$= 2\Delta \hat{k}$ where

Δ is the area of the triangle ΔAOB.

By symmetry, the sum of the moments of all the forces $= 3(2\Delta)\hat{k} = 6\Delta\hat{k}$.

$\therefore$ The system reduces to a couple of moment 6Δ, which is the area of the hexagon also.

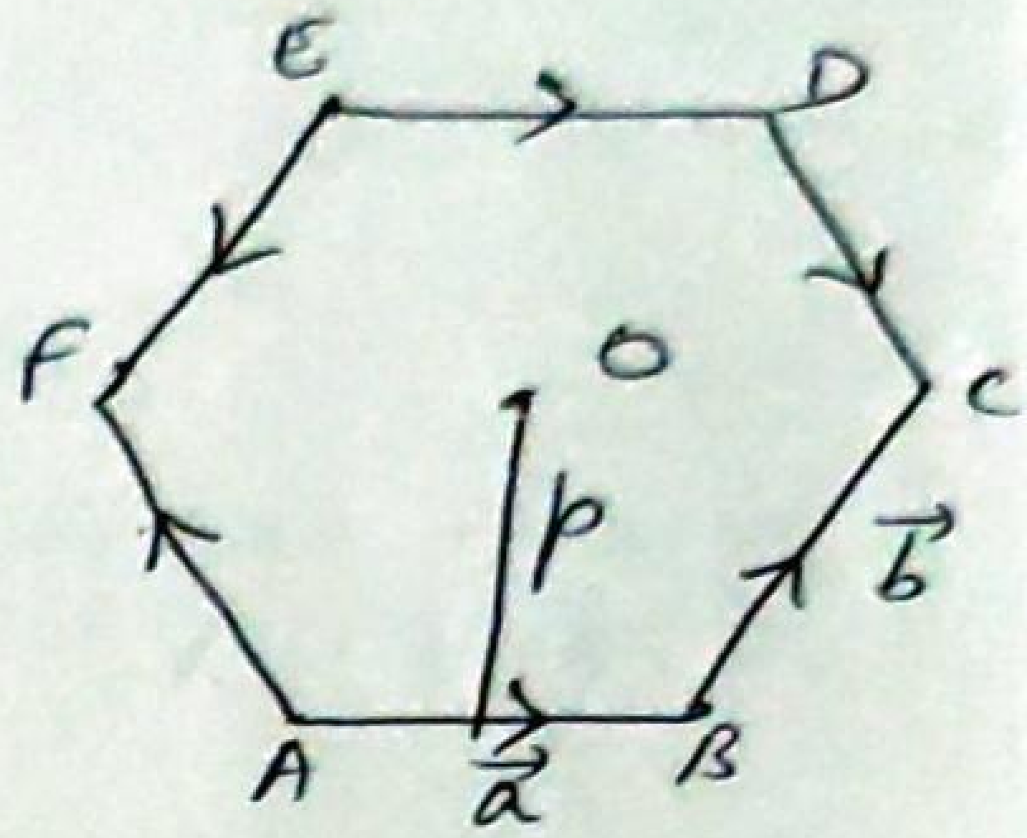
Problems: (B2)

ABCDEF is a regular hexagon. Forces $\hat{AB}$, $2\hat{BC}$, $3\hat{DC}$, $2\hat{ED}$, $5\hat{EF}$, $6\hat{AF}$ act at A, B, C, D, E, F respectively. Show that they are equivalent to a couple and find its moment.

[OR]

ABCDEF is a regular hexagon. Forces $P, 2P, 3P, 2P, 5P, 6P$ act along AB, BC, DC, ED, EF, AF. Show that the six forces are equivalent to a couple and find the moment of the couple.

Solution:



Let the side of the hexagon $= a$.

Let $\overline{AB} = \bar{a}$, $\overline{BC} = \bar{b}$.

Then $\overline{DC} = \overline{DO} + \overline{OC} = \bar{a} - \bar{b}$.

Similarly $\overline{ED} = \bar{a}$, $\overline{EF} = -\bar{b}$,

$\overline{AF} = -\bar{a} + \bar{b}$.

All these six vectors are of length a.

To get the respective unit vectors we have

$$\hat{a} = \frac{\bar{a}}{a}, \quad \hat{b} = \frac{\bar{b}}{a} \ldots$$

$\therefore$ The sum of the given forces $=$

$$= P \cdot \frac{\bar{a}}{a} + 2P \cdot \frac{\bar{b}}{a} + 3P \cdot \frac{\bar{a} - \bar{b}}{a} + 2P \frac{\bar{a}}{a} + 5P \frac{(-\bar{b})}{a} + 6P \cdot \frac{(-\bar{a} + \bar{b})}{a}$$

$= 0$.

$\Rightarrow$ The system reduces to a couple or its equilibrium.

If p is the length of the perpendicular from the centre O to the sides then the scalar sum of the moments of the forces about O

$$= p \cdot [P + 2P - 3P - 2P + 5P - 6P].$$

$$= -3p \cdot P = -3 \frac{\sqrt{3}}{2} \cdot aP. \quad \left[\text{height of the triangle} = \frac{\sqrt{3}}{2} a\right].$$

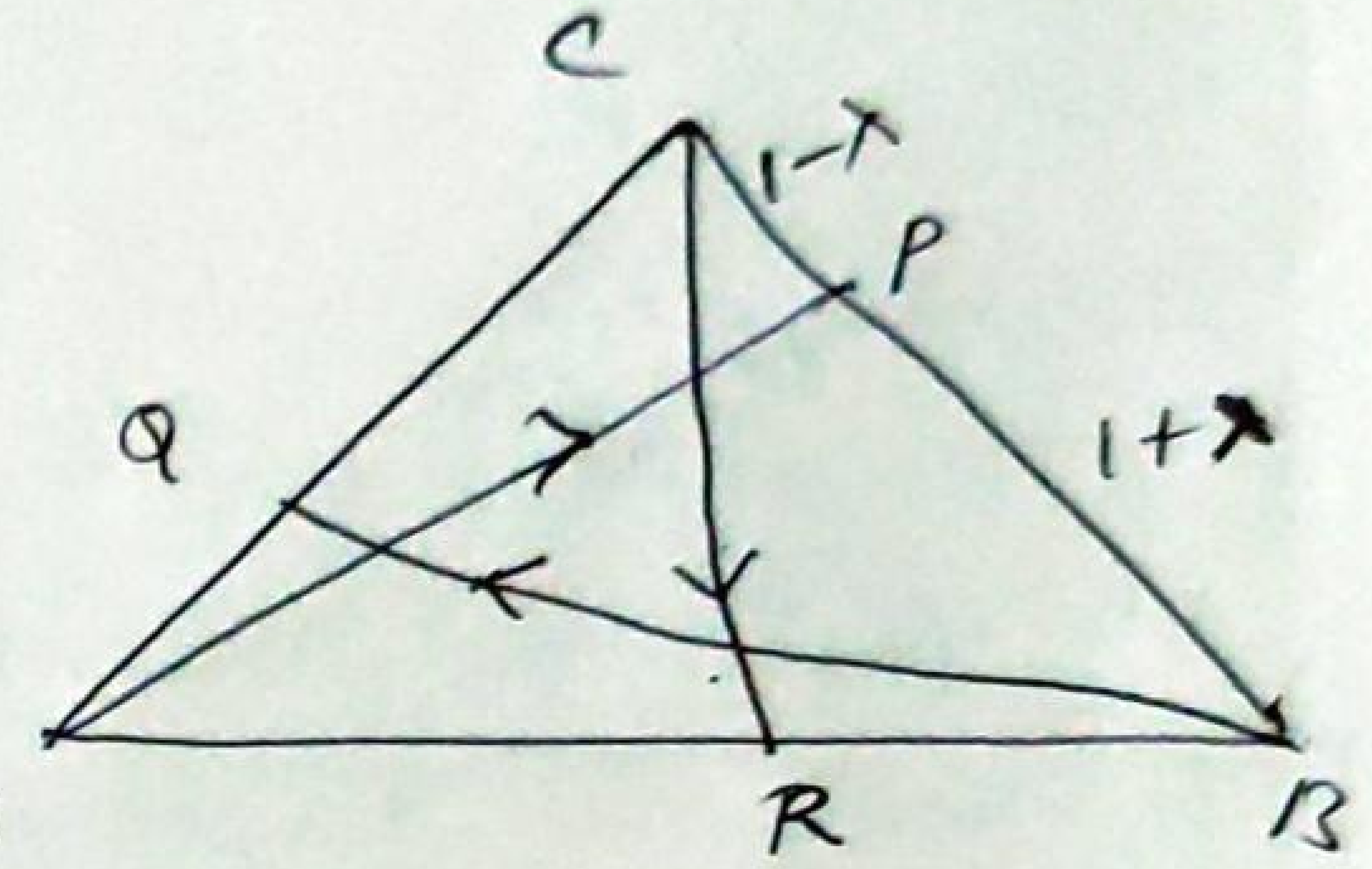
$\therefore$ The system reduces to a couple whose moment $= -\frac{3\sqrt{3}}{2} a P$.

Problem: (B3)

P, Q, R are points on the sides BC, CA, AB of a triangle ABC, dividing them internally in the same ratio $1+\lambda : 1-\lambda$. Show that the forces $\overline{AP}, \overline{BQ}, \overline{CR}$ acting at A, B, C are equivalent to a couple of moment $2\lambda\Delta$, where Δ is the area of the triangle ABC.

Solution:

P divides BC in the ratio $1+\lambda : 1-\lambda$.



$$\therefore \overline{AP} = \frac{(1-\lambda)\overline{AB} + (1+\lambda)\overline{AC}}{(1+\lambda)+(1-\lambda)}$$

$$= \frac{(1-\lambda)}{2}\overline{AB} - \frac{(1+\lambda)}{2}\overline{CA}$$

Similarly, $\overline{BQ} = \frac{1-\lambda}{2}\cdot\overline{BC} - \frac{(1+\lambda)}{2}\overline{AB}$,

and $\overline{CR} = \frac{1-\lambda}{2}\overline{CA} - \frac{1+\lambda}{2}\cdot\overline{BC}$.

Since $\overline{AB}+\overline{BC}+\overline{CA} = 0$, sum of the forces $\overline{AP}+\overline{BQ}+\overline{CR} = 0$.

$\Rightarrow$ Either the forces are in equilibrium (or) they reduce to a couple.

If $\overline{G}$ is the sum of the moments of the forces, say A,

then $\overline{G} = \overline{0} + \overline{AB}\times\overline{BQ} + \overline{AC}\times\overline{CR}$.

$$= \overline{AB}\times\left[\frac{1-\lambda}{2}\overline{BC} - \frac{(1+\lambda)}{2}\overline{AB}\right] + \overline{AC}\times\left[\frac{1-\lambda}{2}\overline{CA} - \frac{1+\lambda}{2}\cdot\overline{BC}\right].$$

$$= \frac{1-\lambda}{2}\left(\overline{AB}\times\overline{BC}\right) - \frac{(1+\lambda)}{2}\left(\overline{AC}\times\overline{BC}\right)$$

$$= \frac{1-\lambda}{2}\cdot 2\Delta\hat{n} - \frac{(1+\lambda)}{2}2\Delta\hat{n} = -2\lambda\Delta\hat{n} \left[\because \overline{AB}\times\overline{BC} = 2\Delta\hat{n}\right]$$

$\Rightarrow$ The forces reduce to a couple of moment $2\lambda\Delta$.

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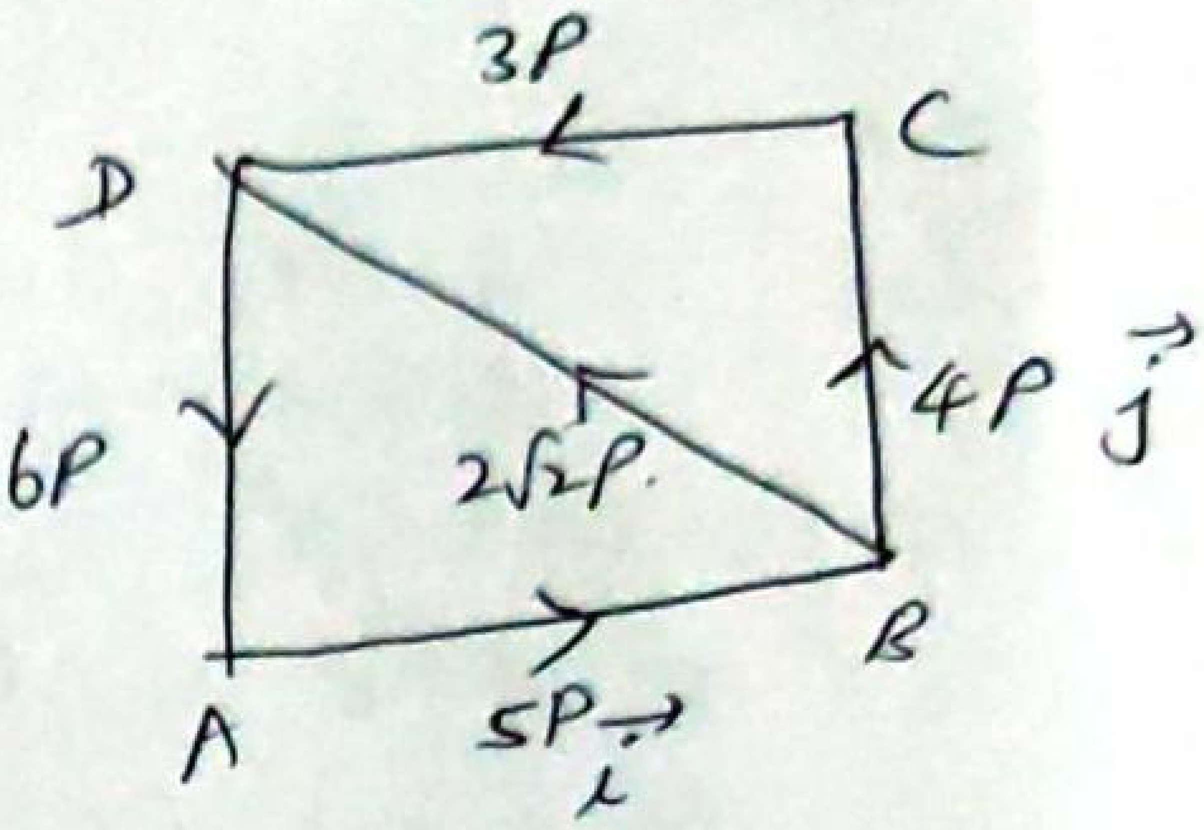
Problem (B4)

ABCD is a square of side a. Forces 5P, 4P, 3P, 6P, $2\sqrt{2}P$ act along AB, BC, CD, DA, BD respectively. Show that the system reduces to a couple of moment 9aP.

Solution:
Let $\vec{i}$ and $\vec{j}$ be unit vectors along AB, BC.



AB = BC = CD = DA = a

Then the forces are

$\overline{AB} = 5P\vec{i}$

$\overline{BC} = 4P\vec{j}$

$\overline{CD} = -3P\vec{i}$

$\overline{DA} = -6P\vec{j}$

$\overline{BD} = \overline{BC} + \overline{CD} = 2\sqrt{2}P\cdot\left(\frac{-\vec{i}+\vec{j}}{\sqrt{2}}\right)$.

Their sum $= \overline{0}$.

$\therefore$ It reduces to a couple.

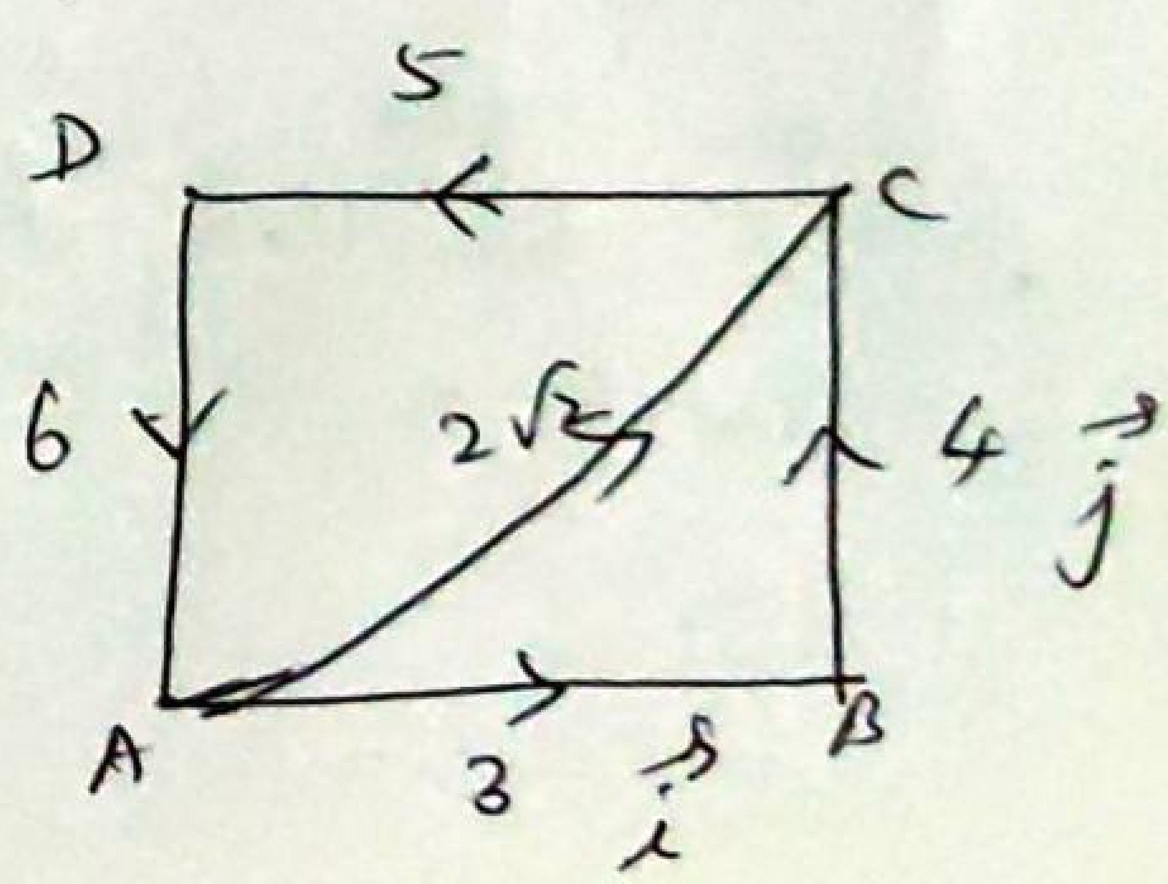
Moment about B $= a(3P) + a(6P)$.

$= 9aP$.

Problem (B5)

Forces of 3, 4, 5, 6 and $2\sqrt{2}$ act along AB, BC, CD, DA and AC of a square ABCD of side a. Show that the system reduces to a couple of moment 9aP.

Solution: Let $\vec{i}$ and $\vec{j}$ be unit vectors along AB, BC.



Given: AB = BC = CD = DA = a.

$\therefore$ Forces are $\overline{AB} = 3\vec{i}$, $\overline{BC} = 4\vec{j}$, $\overline{CD} = -5\vec{i}$

$\vec{DA} = -6\vec{j}$, $\overline{AC} = \frac{\vec{i}+\vec{j}}{\sqrt{2}}$.

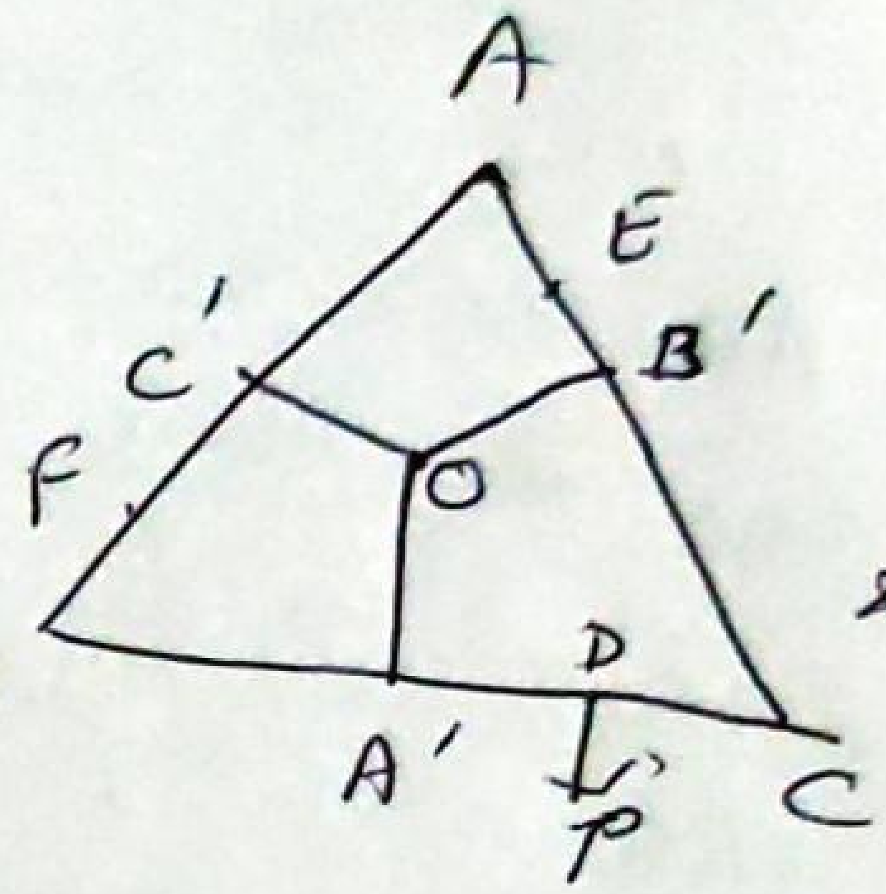
Their sum is $= \overline{0} \Rightarrow$ It reduces to a couple.

$\therefore$ Sum of moments of forces about A

$= a(4P) + a(5P) = 9aP$.

Problem (B6)

ABC is an equilateral triangle of side a. D, E, F divides the sides BC, CA, AB respectively in the ratio 2:1. Three forces, each of magnitude P, act at D, E, F perpendicularly to the sides and outward from the triangle. Prove that they are equivalent to a couple of moment $\frac{1}{2}Pa$.



BD : DC = 2 : 1

CE : EA = 2 : 1

& AF : FB = 2 : 1

Solution:

Let O be the orthocentre and circumcentre of the equilateral Δ^{le},

Let A', B', C' be the mid point of the sides BC, CA and AB.

Now OA' ⊥r BC.

using the theorem " A force acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body together with a couple",

the force P acting at D ⊥r to BC is equivalent to a parallel force P acting at O along OA' together with a couple whose

$$\text{moment} = P \cdot A'D = P \cdot (A'C - DC)$$

$$= P\left(\frac{a}{2} - \frac{a}{3}\right) = \frac{Pa}{6}.$$

Similarly the force P acting at E ⊥r to CA is replaced by a parallel force P acting at O along OB' together with a couple whose moment $= \frac{Pa}{6}$

and the force P acting at F ⊥r to AB is replaced by a parallel force P acting at O along OC' together with a couple whose moment $= \frac{Pa}{6}$.

The three equal forces P acting at O ⊥r to the sides of the triangle are in equilibrium by the perpendicular triangle of forces.

The 3 couples have the same moment $\frac{Pa}{6}$ each in the same direction are equivalent to a single couple.

Moment of the couple $= 3 \times \frac{Pa}{6}$.

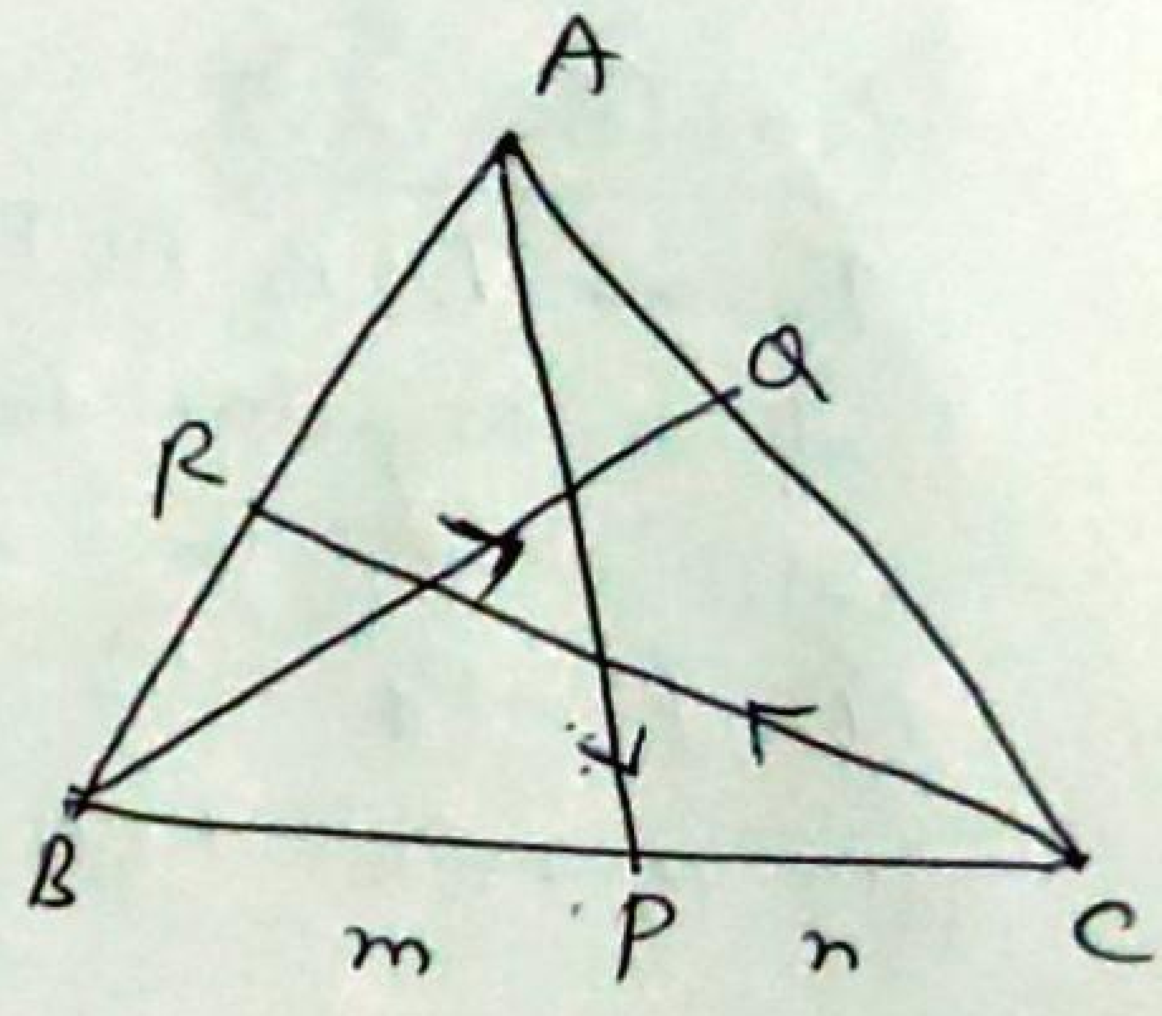
$= \frac{Pa}{2}$.

Problem (37)

P, Q, R are points on the sides BC, CA, AB of a triangle ABC, dividing them internally in the same ratio m:n. Show that the forces $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$ acting at A, B, C are equivalent to a couple of moment $\frac{2(n-m)}{m+n}\Delta$, where Δ is the area of the triangle ABC.

Solution:

Given P, Q, R divides the sides BC, CA, AB of a triangle ABC in the ratio m:n.



(ie) $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB} = \frac{m}{n}$.

Now $\frac{BP}{PC} = \frac{m}{n}$

$\Rightarrow n\,\overline{BP} = m\,\overline{PC}$.

Now by Lami's theorem, $n\overline{AB} + m\overline{AC} = (m+n)\overline{AP}$.

(ie) $\overline{AP} = \frac{n}{m+n}\overline{AB} + \frac{m}{m+n}\overline{AC} \longrightarrow (1)$

Similarly $\overline{BQ} = \frac{n}{m+n}\overline{BC} + \frac{m}{m+n}\overline{BA} \longrightarrow (2)$

and $\overline{CR} = \frac{n}{m+n}\overline{CA} + \frac{m}{m+n}\overline{CB} \longrightarrow (3)$

Adding (1), (2) and (3), we get

$$\overline{AP} + \overline{BQ} + \overline{CR} = \frac{n}{m+n}\left(\overline{AB}+\overline{BC}+\overline{CA}\right) + \frac{m}{m+n}\left(\overline{AC}+\overline{BA}+\overline{CB}\right)$$

$$= \frac{n-m}{m+n}\left(\overline{AB}+\overline{BC}+\overline{CA}\right).$$

Now Consider the three forces represented Completely by $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.

The vector sum $= 0$.

$\Rightarrow$ The forces are equivalent to a Couple.

Taking moments about A, the moment of this

Couple $= BC \cdot AD$

$= 2\Delta$, AD being the altitude through A.

$\left[\because \text{Area of a triangle} = \frac{1}{2}bh\right]$

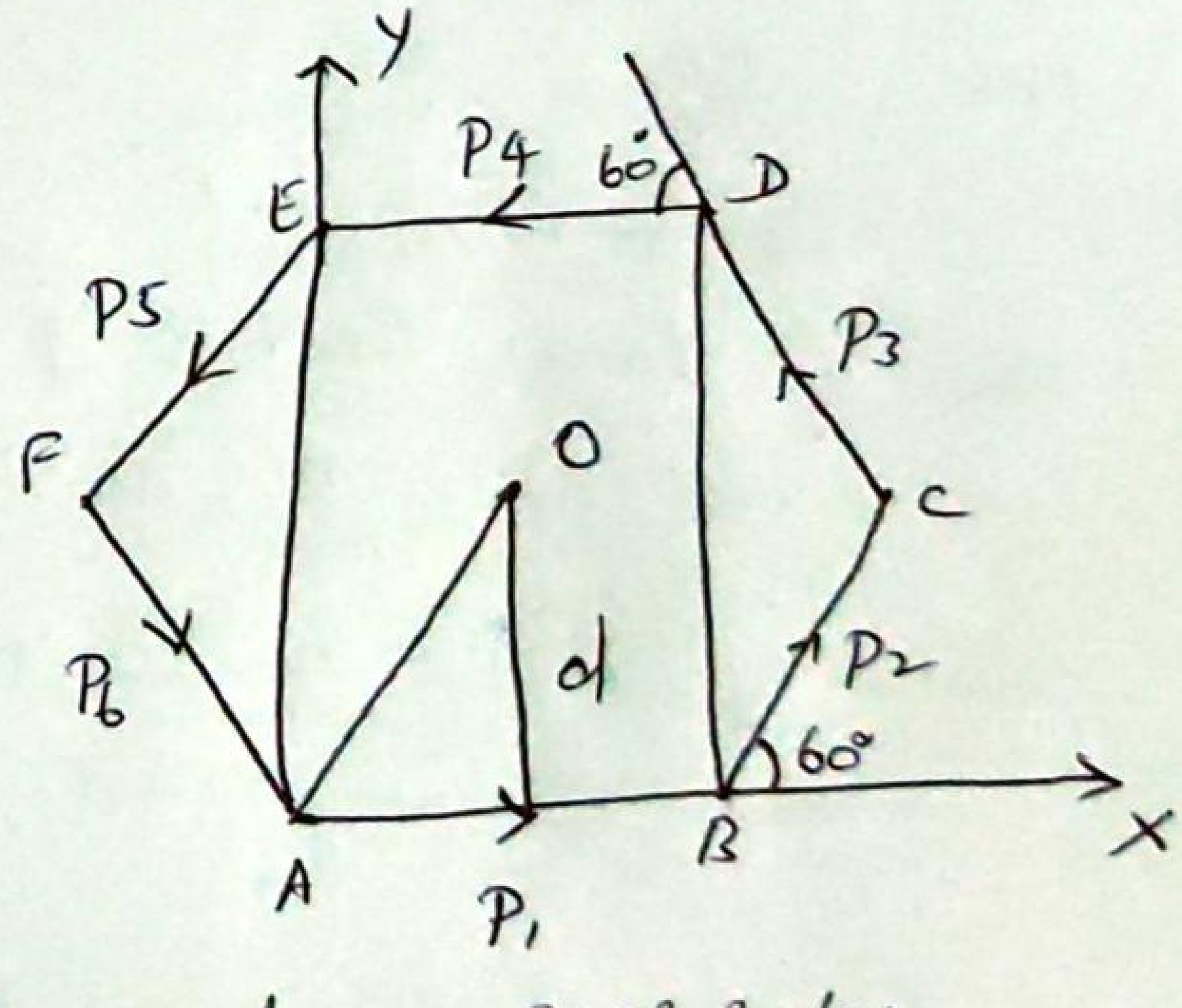
Hence $\overline{AP} + \overline{BQ} + \overline{CR} = 2\left(\frac{n-m}{m+n}\right)\Delta.$

Problem (88)

Forces $P_1, P_2, P_3, P_4, P_5, P_6$ acts along AB, BC, CD, DE, EF, FA of a regular hexagon taken in order. Show that they will be in equilibrium if

(i) $P_1 - P_4 = P_3 - P_6 = P_5 - P_2$.

(ii) $P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 0$.



Solution:

Let O be the centre of the regular hexagon ABCDEF.

For equilibrium, the algebraic sum of the moments of the forces about $O = 0$.

$\therefore$ ~~$P_1d + P_2d + P_3d + P_4 + P_5 + P_6$~~

$(P_1 + P_2 + P_3 + P_4 + P_5 + P_6) \cdot d = 0.$

where d is the distance of O from each sides.

$\therefore\ P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 0 \longrightarrow (1).$

Take AB as X axis

and AE as Y axis.

Resolving along AB, we get

$F_1 + F_2 \cos 60^\circ - F_3 \cos 60^\circ - F_4 - F_5 \cos 60^\circ + F_6 \cos 60^\circ = 0.$

$\Rightarrow (F_1 - F_4) + \frac{1}{2}(F_2 - F_3 - F_5 + F_6) = 0 \longrightarrow (2)$

Resolving along AE, we get

$F_2 \sin 30^\circ + F_3 \cos 30^\circ - F_5 \cos 30^\circ - F_6 \cos 30^\circ = 0.$

(ie) $F_2 + F_3 - F_5 - F_6 = 0 \longrightarrow (3).$

$(3) \Rightarrow F_3 - F_6 = F_5 - F_2 \rightarrow (4).$

using (4) in (2), we get

$(F_1 - F_4) + \frac{1}{2}\left[F_2 - F_5 - (F_3 - F_6)\right] = 0.$

$\Rightarrow F_1 - F_4 + \frac{\left[F_2 - F_5 - (F_5 - F_2)\right]}{2} = 0.$

$\Rightarrow F_1 - F_4 + F_2 - F_5 = 0$

$\Rightarrow \boxed{F_1 - F_4 = F_5 - F_2} \longrightarrow (5).$

Combining (4) and (5), we get

$F_1 - F_4 = F_5 - F_2 = F_3 - F_6. \longrightarrow (6)$

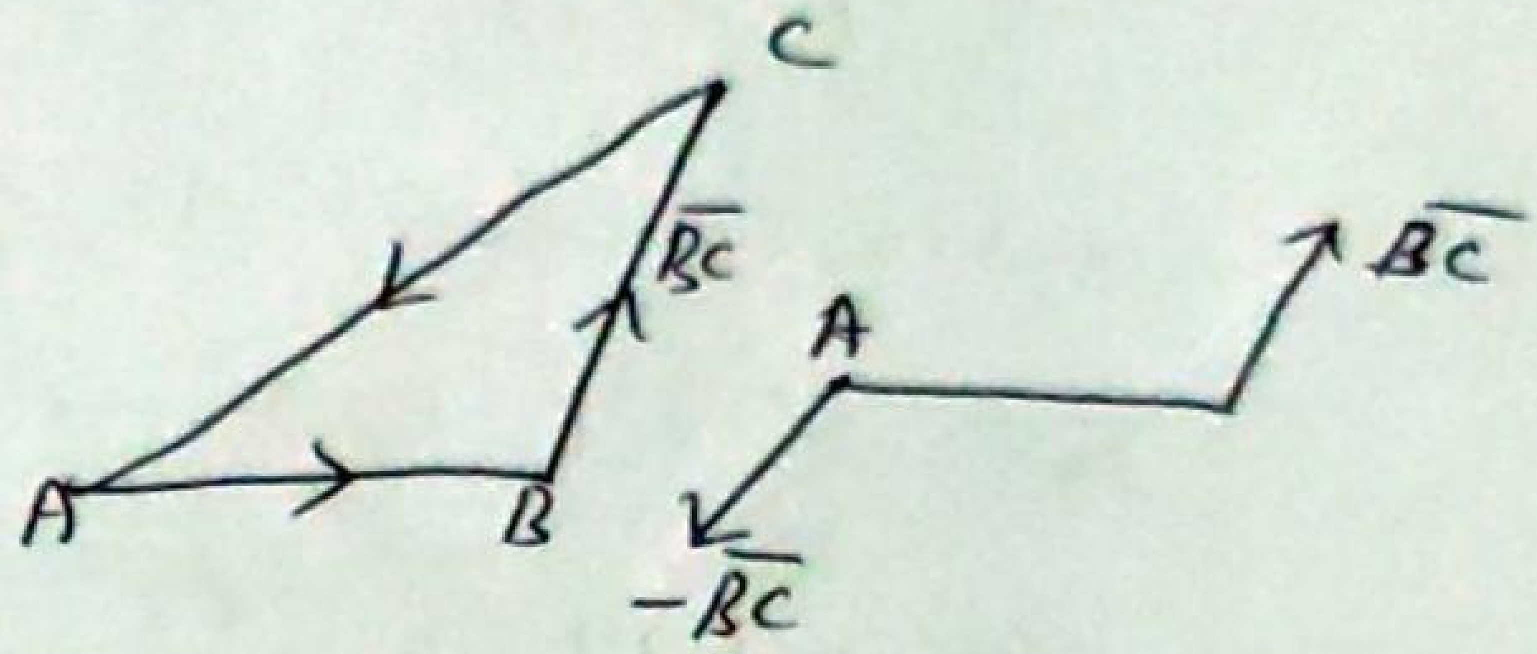
Hence for equilibrium, the necessary conditions are expressed by equations (1) and (6).

(C) Moment of a certain Couple as an area

Bookwork : (3)

Three Coplanar forces represented by and acting along the sides of a triangle, taken in order, reduce to a couple, the magnitude of whose moment being equal to twice the area of the triangle.

Proof:



Let ABC be the given triangle.
Then the given forces are
$\overline{AB}$ acting at A,
$\overline{BC}$ acting at B,
$\overline{CA}$ acting at C.
The first and third forces act at A.
Their resultant is $\overline{AB}+\overline{CA}$ ($=$ $\overline{CA}+\overline{AB}$

$$= \overline{CB}$$
$$= -\overline{BC} \text{ acting at } A.$$

So the given 3 forces are equivalent to the two forces,
(i) $-\overline{BC}$ acting at A,
(ii) $\overline{BC}$ acting at B.
(e) the given forces are equivalent to the
Couple $(-\overline{BC}, \overline{BC})$.
Its moment is $\overline{AB}\times\overline{BC}$.
But we know that,
the area of the triangle $ABC = \frac{1}{2}\left|\overline{AB}\times\overline{BC}\right|$.
Hence the magnitude of the moment of the couple is $|\overline{AB}\times\overline{BC}|$.

$$= 2\,(\text{area of the triangle } ABC).$$

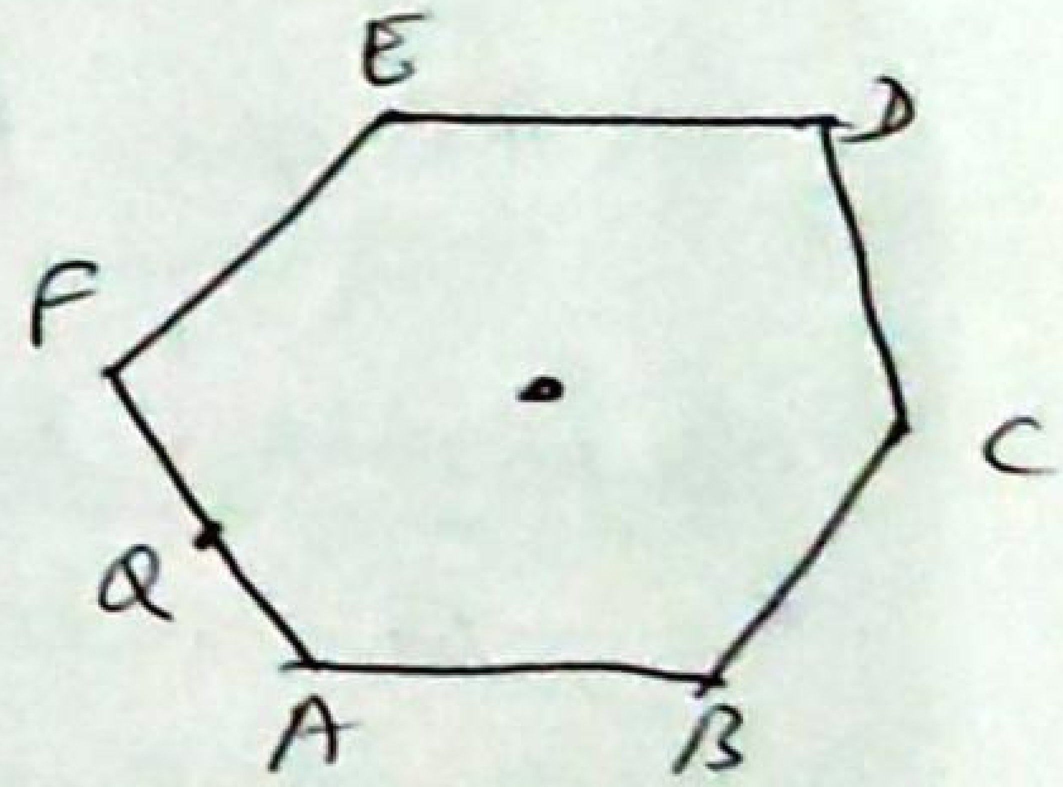
An extension:

The above bookwork can be extended to any number of coplanar forces acting on a rigid body,

(ii) n coplanar forces represented by and acting along the sides of a polygon $A_1, A_2, \ldots A_n$ taken in order, reduce to a couple, the magnitude of whose moment is equal to twice the area of the polygon.

Problem (C1)

Five equal forces act along the sides AB, BC, CD, DE, EF of a regular hexagon. Show that the sum of moments of these forces about any point Q on FA is a constant.



Solution:

Let the side of the hexagon = a and the equal forces = P.

Now the 6 forces $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$ acting at A, B, C, D, E, F reduces to a couple of moment which is = Area of the hexagon.

$$= 6 \cdot \frac{1}{2} \cdot a \cdot a \cdot \sin 60^\circ$$

$$= \frac{3\sqrt{3}}{2} \cdot a^2.$$

$\therefore$ Taking the moments of these six forces about Q

$$\overline{QA} \times \overline{AB} + \overline{QB} \times \overline{BC} + \cdots + \overline{QF} \times \overline{FA} = \frac{3\sqrt{3}}{2} a^2 \hat{k} \longrightarrow (1)$$

where $\hat{k}$ is the unit vector perpendicular to the plane of the hexagon.

Dividing both sides of ① by a and multiplying by P, we get

$$\overline{QA} \times (P\,\hat{AB}) + \overline{QB} \times (P\,\hat{BC}) + \cdots + \overline{QF} \times (P \cdot \hat{FA}) = \frac{3\sqrt{3}}{2} a P . \hat{k} \quad \longrightarrow ②$$

In the sixth term of ②, $\vec{QF}$ and $\hat{FA}$ are parallel and so it vanishes.

(ie) $\overline{QF} \times (P \cdot \hat{FA}) = \overline{0}$

$$\therefore \overline{QA} \times (P\,\hat{AB}) + \overline{QB} \times (P\,\hat{BC}) + \cdots + \vec{QE} \times (P\,\hat{EF}) = \frac{3\sqrt{3}}{2} a P \hat{k}.$$

$\therefore$ Sum of the scalar moments of the given five forces about Q $= \frac{3\sqrt{3}}{2} a P$ = a Constant.

<u>Couples in a parallel plane.</u>

The vector moment of a couple is perpendicular to the plane of the couple.

So, if two couples have parallel vector moments, then they should be in the same plane or in parallel planes.

<u>Book work : 4</u>

Two Couples of equal moments are equivalent.

<u>Proof:</u> Let (P, p) and (Q, q) be two couples in one plane having the same equal moments in magnitude and direction.

The vector sum of the constituent forces of each couple = 0.
So the couples satisfy the first condition of equivalence.
Since the moment of a couple is = sum of the moments of the constituent forces about any point, by hypothesis, the couples satisfy the second condition of equivalence.
$\therefore$ The couples are equivalent.

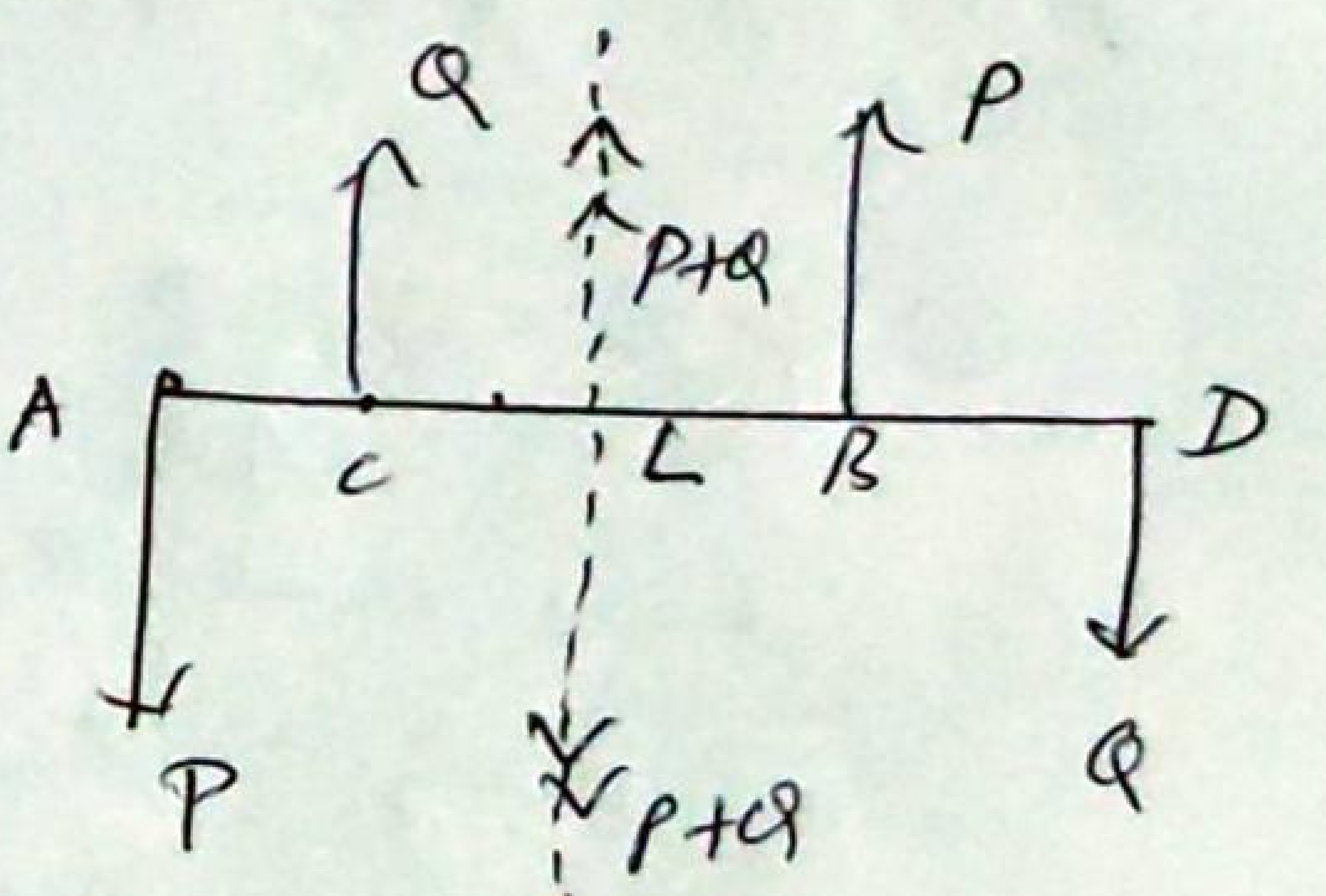
Bookwork: (5)

Two coplanar couples whose moments are equal in magnitude but opposite in direction keep a rigid body in equilibrium.

Proof: Let (P, p) and (Q, q) be two given couples such that $Pp = Qq$ in magnitude but opposite in sign.

Case 1: Let the forces P and Q be parallel.

Draw a straight line $\perp$r to the lines of action of the forces, meeting them at A, B, C, D.



Since the moments of the couples are equal, we have

$$P \cdot AB = Q \cdot CD \rightarrow (1)$$

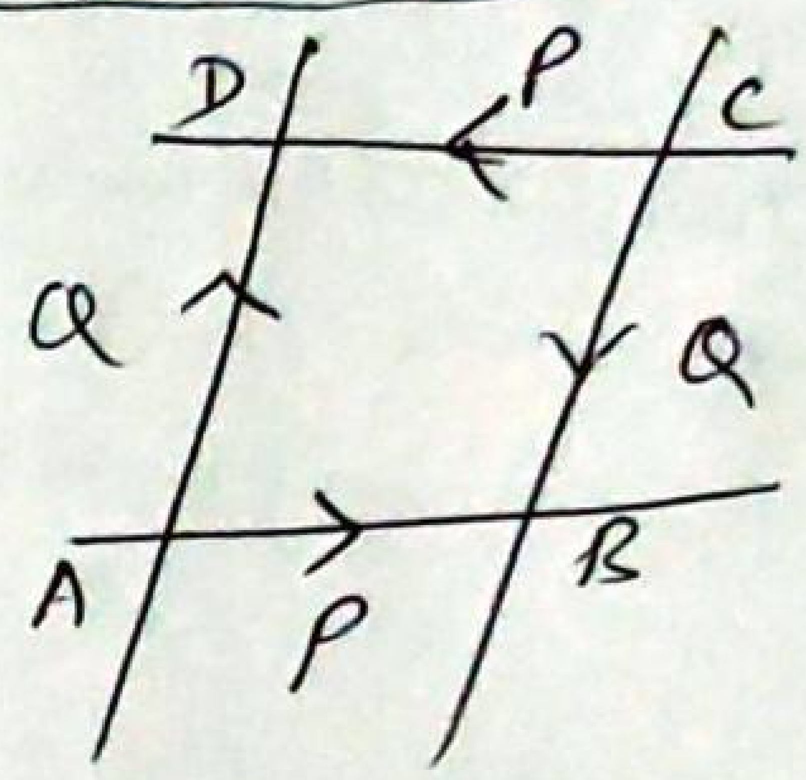
The downward like parallel forces P at A and Q at D can be compounded into a single force P+Q acting at L such that $P \cdot AL = Q \cdot DL \rightarrow (2)$

$(1) - (2) \Rightarrow P(AB - AL) = Q(CD - DL)$

$\Rightarrow P \cdot BL = Q \cdot CL \rightarrow (3).$

(3) $\Rightarrow$ the resultant of the upward like parallel forces P at B and Q at C will also pass through L. The magnitude of this resultant is also (P+Q) but it is opposite in direction to the previous resultant. Thus the two resultants balance each other. Hence the 4 forces forming the couples are in equilibrium.

<u>case (ii)</u> <u>Let the forces P and Q intersect.</u>



Let the two forces P of the couple (P, p) meet the two forces Q of the couple (Q, q) at the points A, B, C, D.

clearly ABCD is a parallelogram.

Let AB represent P.

As the moments of the two couples are equal, we have

$$P \cdot p = Q \cdot q \longrightarrow (1)$$

Also $AB \cdot p = AD \cdot q$ (each being equal to the area of the parallelogram ABCD) $\rightarrow (2)$

$$(1) \div (2) \Rightarrow \frac{P}{AB} = \frac{Q}{AD} \longrightarrow (3)$$

(3) shows that the side AD will represent Q in the same way the side AB represents P.

The 2 forces P & Q ~~acting~~ acting at A can be compounded by parallelogram law so that

$(P+Q)$ at $A = \overline{AB} + \overline{AD} = \overline{AC}$

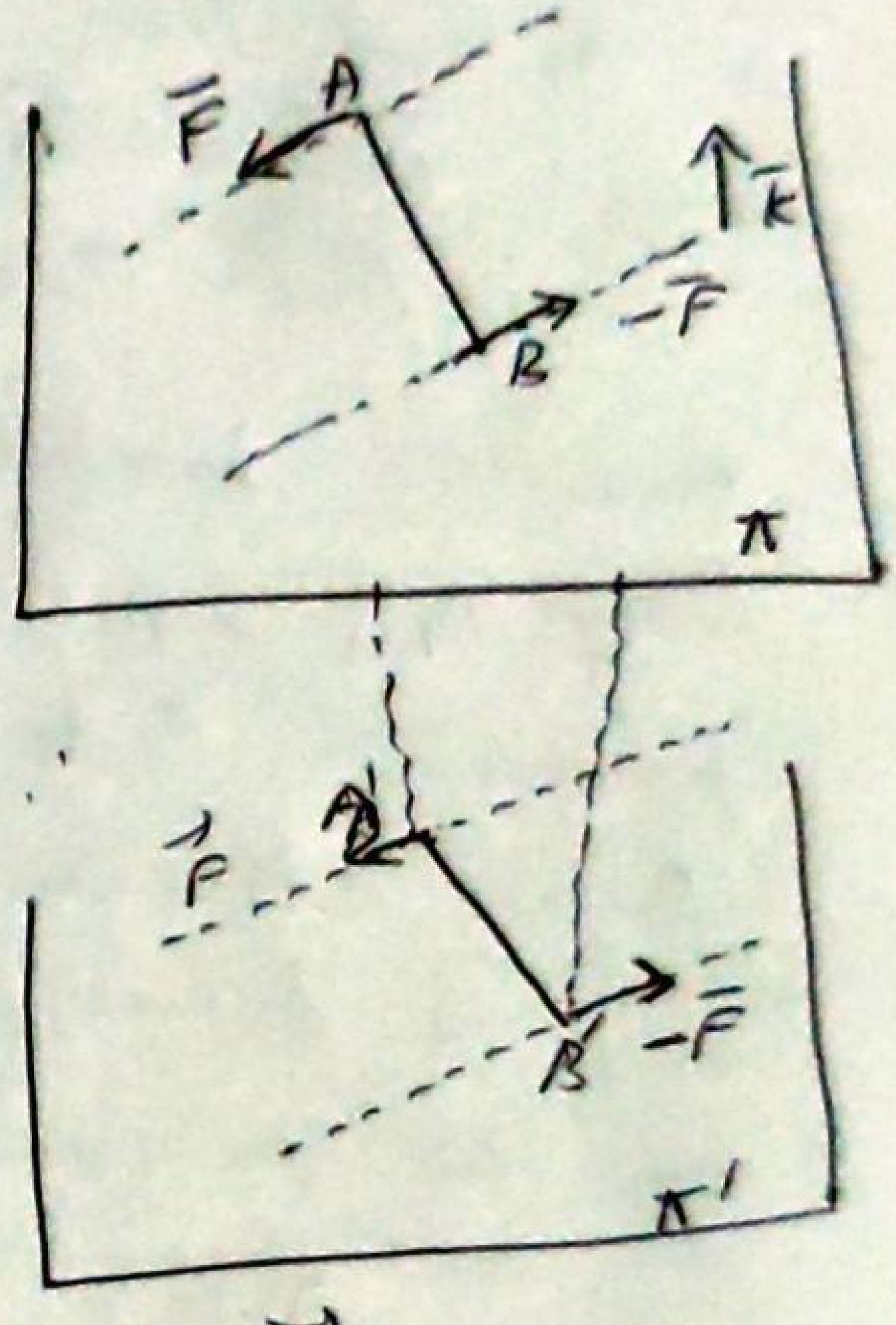
Similarly, $(P+Q)$ at $C = \overline{CD} + \overline{CB} = \overline{CA}$.

The two resultants $\overline{AC}$ and $\overline{CA}$ being equal and opposite cancel each other.

Hence the four forces forming the couples are in equilibrium.

Bookwork: 6

A couple can be transferred to a plane parallel to its own plane without altering its effect on the rigid body on which it is acting.



Proof:

Suppose ($\vec{F}$ at A, $-\vec{F}$ at B) is a given couple in the plane π ~~AB~~ with AB as its arm.

Let π' be a plane parallel to π and $A'B'$ the projection of AB on π'.

Let $\vec{k}$ be the unit vector perpendicular to π.

Consider the couple ($\vec{F}$ at A', $-\vec{F}$ at B')

The moment of the given couple $= (AB)(F)\vec{k}$

and the other couple is $(A'B')(F)\vec{k}$.

But $AB = A'B'$.

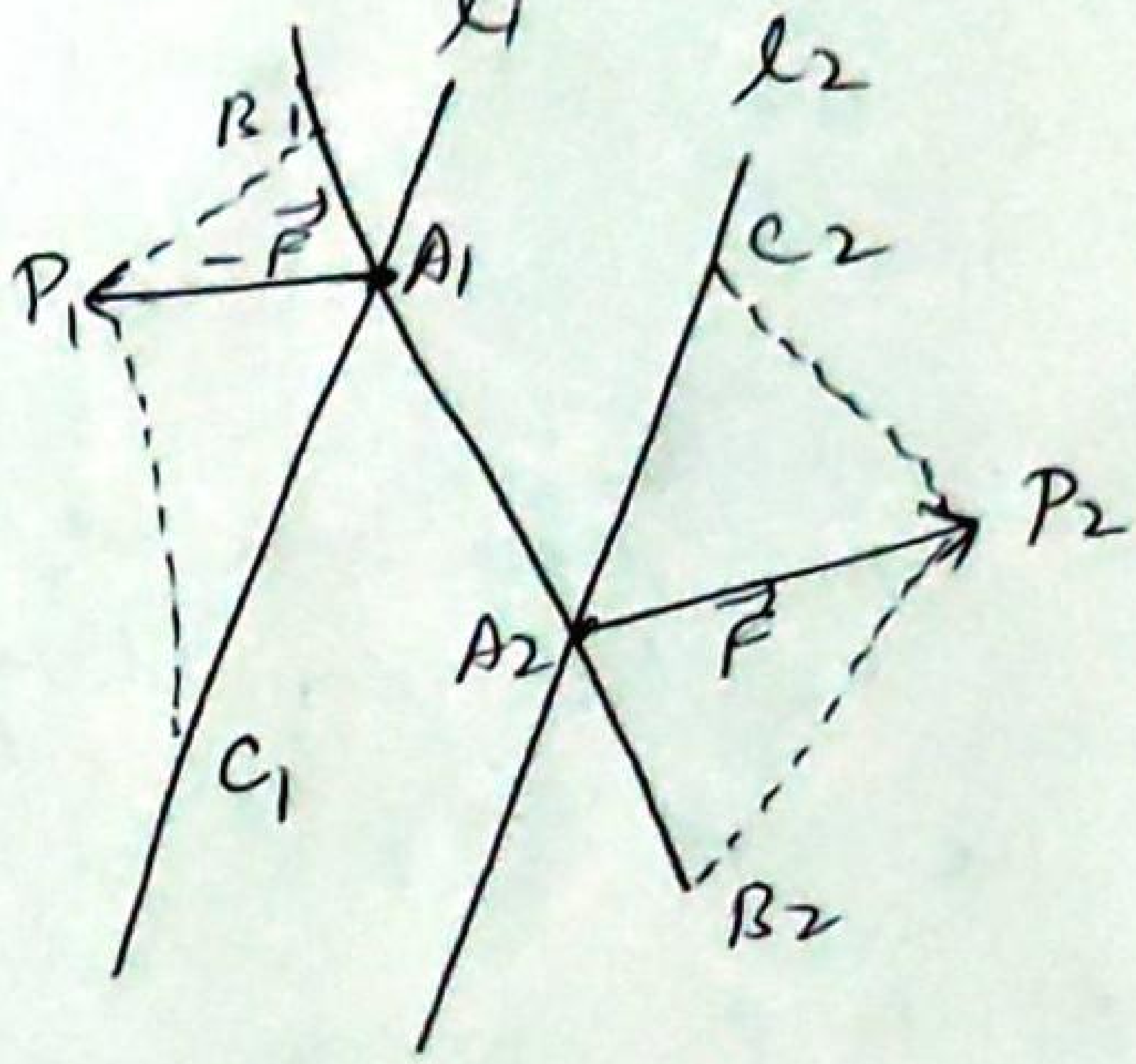
So these moments are equal and hence the couples are equivalent.

Remark: Couples in a set of parallel planes can be compounded into a single couple in a plane parallel to the given planes.

Bookwork: (7)

Given a couple, to find an equivalent couple having its constituent forces along two given parallel lines in the plane of the couple.

Proof:



Let l_1 and l_2 be the given parallel lines.

Let $-\vec{F}$ and $\vec{F}$ be the constituent forces of the given couple.

Let them meet l_1 and l_2 at A_1 and A_2.

Suppose $\overline{A_1P_1} = -\overline{F}$, $\overline{A_2P_2} = \overline{F}$

Draw the parallelograms $A_1B_1P_1C_1$, $A_2B_2P_2C_2$. having A_1P_1, A_2P_2 as diagonals and the sides being parallel to A_1A_2 and l_1.

Since $A_1P_1 = A_2P_2$, the parallelograms are equal.

$\Rightarrow A_1B_1 = A_2B_2$ & $A_1C_1 = A_2C_2$.

Now the forces $-\vec{F}$ at A_1 and $\overline{F}$ at A_2 are equivalent to $\overline{A_1B_1}$, $\overline{A_1C_1}$ at A_1 and $\overline{A_2B_2}$, $\overline{A_2C_2}$ at A_2.

Of these 4 forces, $\overline{A_1B_1}$, $\overline{A_2B_2}$ are equal and opposite and act along the same line.

$\therefore$ They cancel each other.

$\Rightarrow \overline{A_1C_1}$, $\overline{A_2C_2}$ along l_1, l_2 are left out.

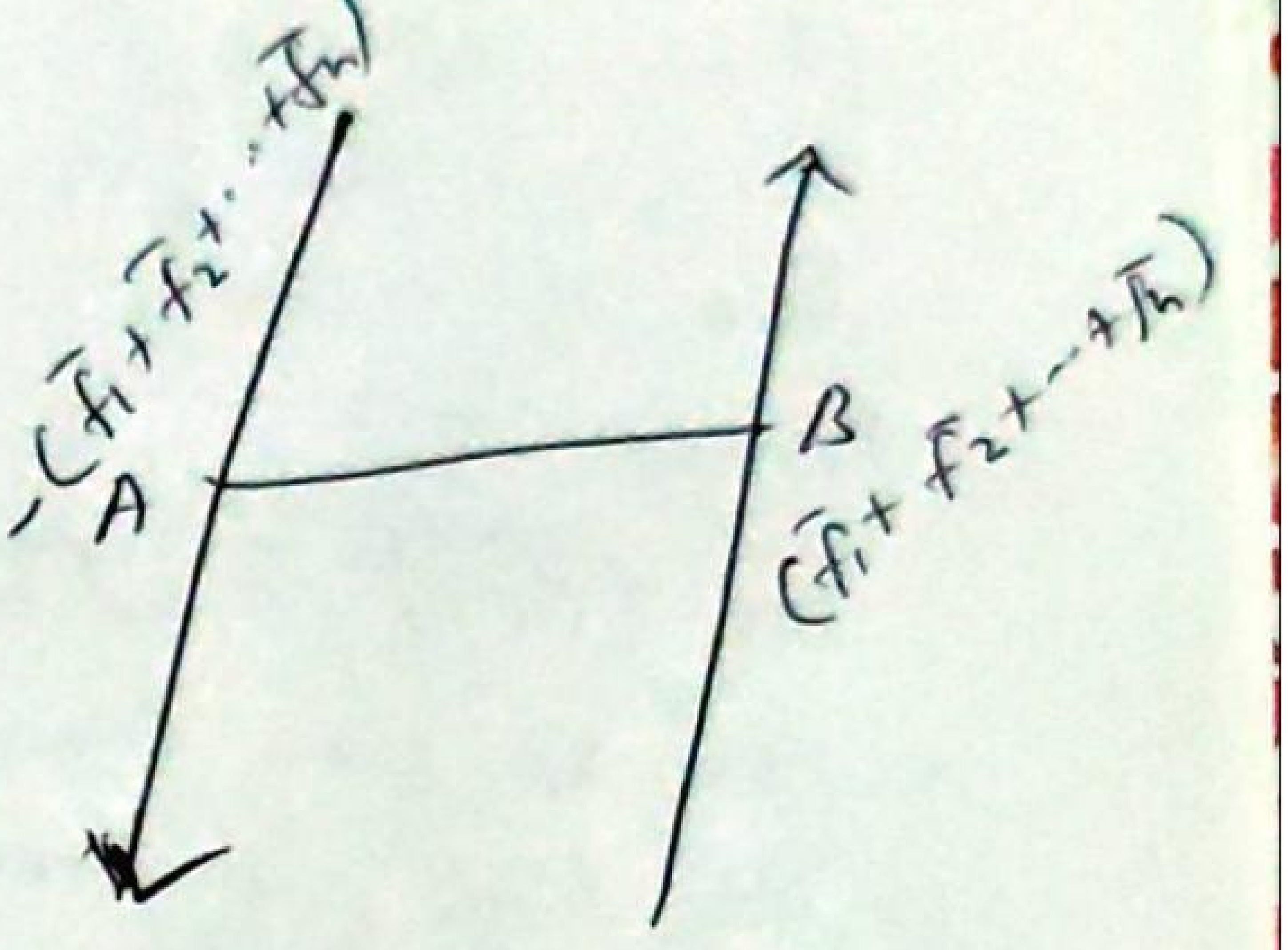
But A_1C_1 and A_2C_2 are equal and opposite.

So they form the couple $(\overline{A_1C_1}, \overline{A_2C_2})$.

Bookwork: 8

A system of coplanar couples acting on a rigid body is equivalent to a couple in the same plane whose moment is equal to the sum of the moments of the given couples.

Proof:



Let $(-\bar{F_1}, \bar{F_1}), (-\bar{F_2}, \bar{F_2}), \dots (-\bar{F_n}, \bar{F_n})$ be n given coplanar couples and let their moments be $\bar{G_1}, \bar{G_2}, \dots, \bar{G_n}$.

Suppose that l_1 and l_2 are two arbitrarily chosen parallel lines in the plane of the couples and that A and B are two points on them.

Let the couples $(-\bar{f_1}, \bar{f_1}), (-\bar{f_2}, \bar{f_2}) \dots (-\bar{f_n}, \bar{f_n})$ having the constituent forces $-\bar{f_1}, -\bar{f_2}, \dots, -\bar{f_n}$ along l_1, and the constituent forces $\bar{f_1}, \bar{f_2}, \dots, \bar{f_n}$ along l_2, be respectively equivalent to the given couples.

Then $\overline{AB} \times \bar{f_1} = \bar{G_1}, \quad \overline{AB} \times \bar{f_2} = \bar{G_2}, \dots, \overline{AB} \times \bar{f_n} = \bar{G_n}$.

Now the resultant of the n forces along l_1 and l_2 are

$$-(\bar{f_1}+\bar{f_2}+\dots+\bar{f_n}), \quad (\bar{f_1}+\bar{f_2}+\dots+\bar{f_n}).$$

They form a couple which is equivalent to the given n couples.

The moment of this couple is

$$\overline{AB} \times (\bar{f_1}+\bar{f_2}+\dots+\bar{f_n}) = \overline{AB} \times \bar{f_1} + \dots + \overline{AB} \times \bar{f_n}$$

$$= \bar{G_1} + \bar{G_2} + \dots + \bar{G_n}$$

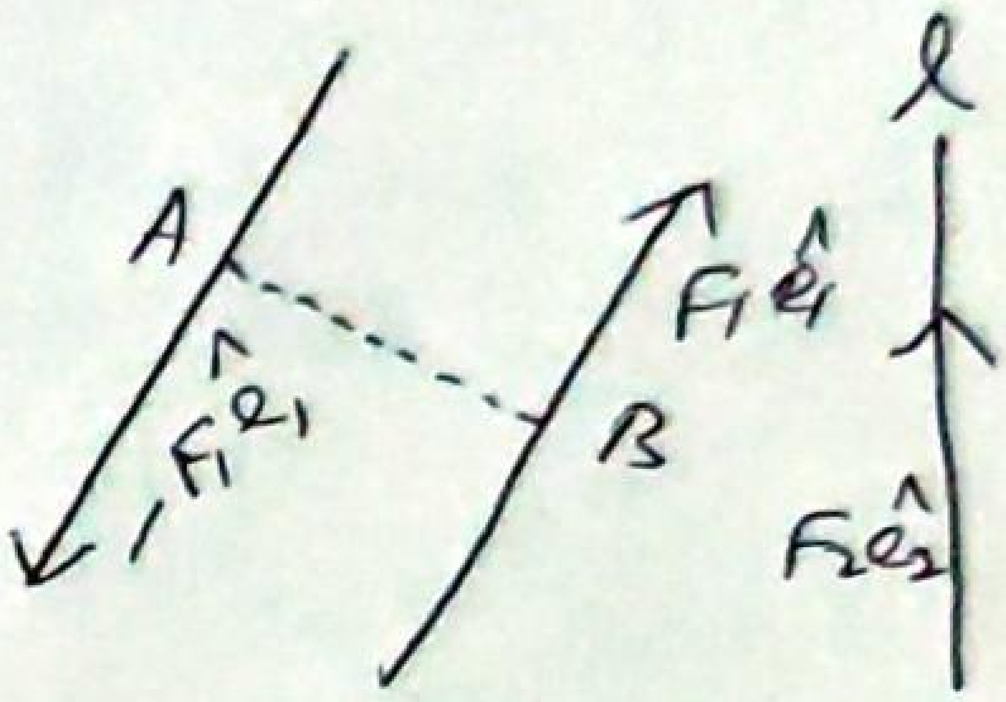
= the sum of the moments of the given couples.

Resultant of a Couple ~~and a force~~ and a force.

Bookwork : (9)

To show that a couple and a force in the same plane reduces to a single force. This single force is the same as the given force, but has a different parallel line of action.

Proof:



Let $(-F_1\hat{e}_1, F_1\hat{e}_1)$ be the given couple and let AB be a common perpendicular to the lines of action of its constituent forces.

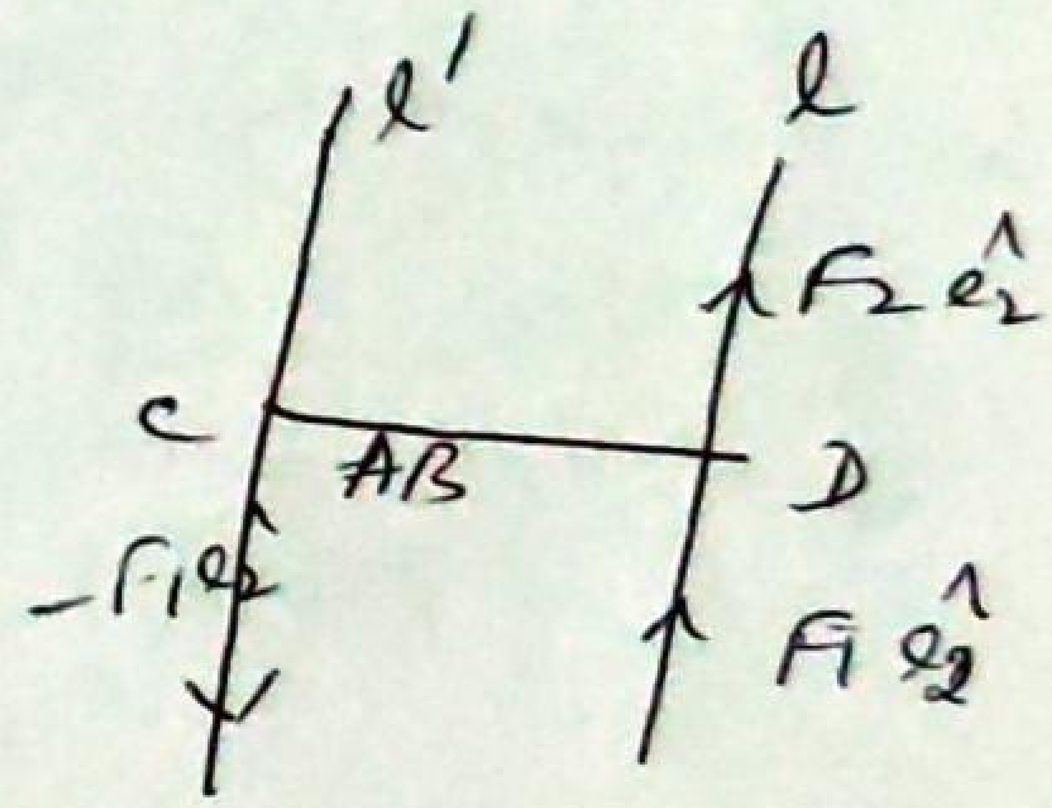
Then its moment $= \overline{AB} \times F_1\hat{e}_1 = AB \cdot F_1 \cdot \hat{n} \rightarrow (1)$.

Let the given force be $F_2\hat{e}_2$ acting along the line l.

Take a line l' parallel to l at a distance AB from l. (2nd diagram).

Suppose CD is a common perpendicular to l' and l.

Then $CD = AB$.



Consider the couple $(-F_1\hat{e}_2, F_1\hat{e}_2)$ having its constituent forces along l' and l.

Its moment is same as (1).

So this new couple $(-F_1\hat{e}_2, F_1\hat{e}_2)$ is equivalent to the given couple.

Hence the given couple and the given force are equivalents to the new couple $(-F_1\hat{e}_2, F_1\hat{e}_2)$ and the force $F_2\hat{e}_2$, acting along l.

So the given couple and the given force are equivalent to the parallel forces.

$(F_1+F_2)\hat{e_2}$, $-F_1\hat{e_2}$ acting along l, l'

Their resultant is $(F_1+F_2)\hat{e_2} + (-F_1\hat{e_2}) = F_2\hat{e_2}$

which is the single given force but acting at the point which divides CD externally in the ratio $(F_1+F_2) : F_1$.

D) Equation of the Line of action of the resultant.

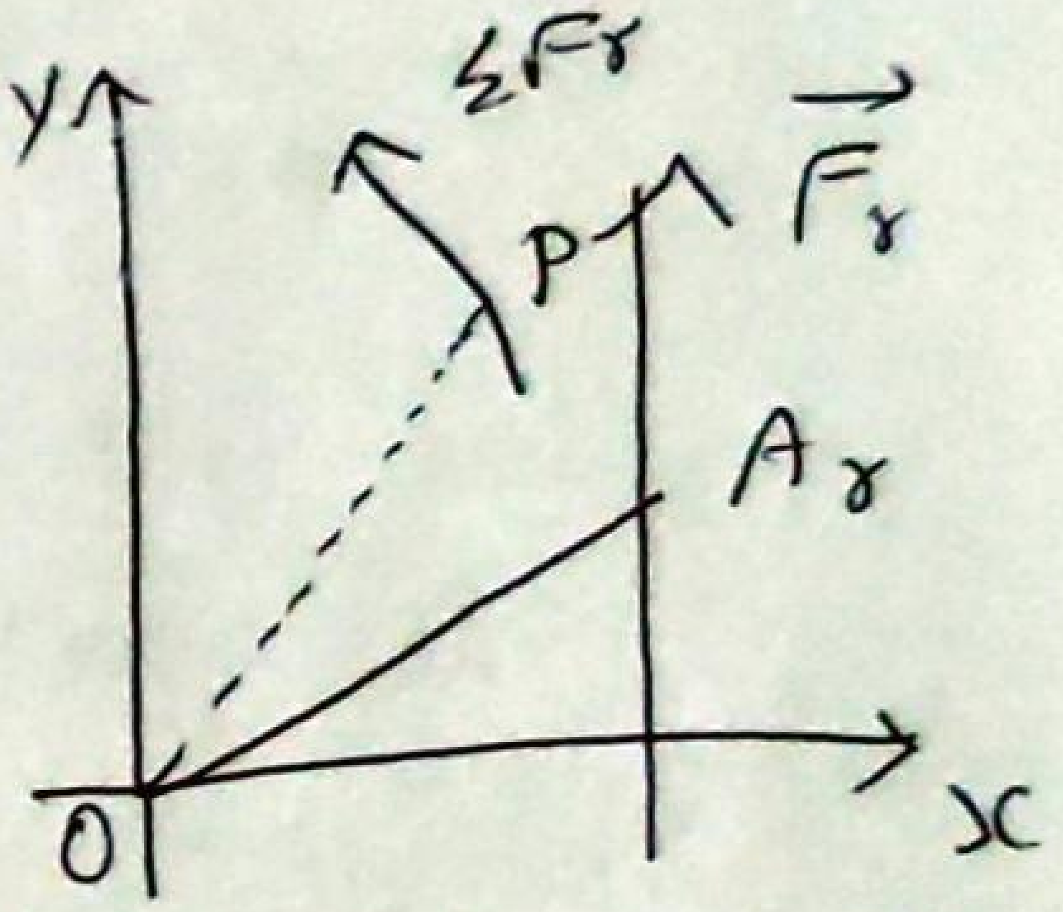
Bookwork : (10)

When a system of coplanar forces $\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}$ acting at $A_1, A_2, \ldots, A_n$, reduce to a single force, to find the equation of its line of action.

Proof :

Choose any two perpendicular lines Ox, Oy in the plane of the forces as the x, y axes.

Let $\vec{i}, \vec{j}$ be the unit vectors in their directions.



Let $P(x, y)$ be any point on the line of action of the resultant force $\Sigma\overline{F_r}$ of the system.

Then any relation in x, y is the equation of the line.

Now $\overline{OP} = x\vec{i} + y\vec{j}$.

Let P_r, Q_r be the components of $\vec{F_r}$ in the $\vec{i}, \vec{j}$ directions.

Then $\overline{F_r} = P_r\vec{i} + Q_r\vec{j}$.

Since the sum of the moments of the forces about any point, say O, equals the moment of the resultant about O,

$$\Sigma(\overline{OA_r} \times \overline{F_r}) = \overline{OP} \times (\Sigma \overline{F_r}).$$

$$\Rightarrow \overline{OP} \times (\Sigma \vec{F_r}) - \Sigma(\overline{OA_r} \times \overline{F_r}) = \overline{0}.$$

(i.e) $(x\vec{i} + y\vec{j}) \times \Sigma(P_r\vec{i} + Q_r\vec{j}) - \Sigma(\overline{OA_r} \times \overline{F_r}) = \overline{0}.$

(i.e) $(x\vec{i} + y\vec{j}) \times [(\Sigma P_r)\vec{i} + (\Sigma Q_r)\vec{j}] - \Sigma(\overline{OA_r} \times \overline{F_r}) = \overline{0}.$

(i.e) $x(\Sigma Q_r)\vec{k} - y(\Sigma P_r)\vec{k} - (\Sigma p_r F_r)\vec{k} = \vec{0}.$

p_r is the perpendicular distance of O from $\vec{F}$.

Then the equation of the line of action of the resultant is $(\Sigma Q_r)x - (\Sigma P_r)y - \Sigma p_r F_r = 0.$

(or) $(\Sigma Q_r)x - (\Sigma P_r)y - \Sigma G_r = 0 \longrightarrow$ (1).

where $G_r = p_r F_r$.

The equation can be put in the form

$$Yx - Xy - G = 0 \longrightarrow (2)$$

where $X = \Sigma P_r$ = sum of the components of the forces in x direction.

$Y = \Sigma Q_r$ = sum of the components of forces in y direction.

$G = \Sigma G_r = \Sigma p_r F_r$

= sum of the scalar moments of the forces about the origin.

Now, we have the resultant force is

$$\overline{F_1} + \overline{F_2} + \dots + \overline{F_n} = X\vec{i} + Y\vec{j}.$$

whose magnitude is $\sqrt{X^2+Y^2}$

and the line of action is $Yx - Xy = G$,

The slope of the line is Y/X.

Note:

(2) can be written as

$$\begin{vmatrix} X & Y \\ x & y \end{vmatrix} + G = 0$$

where (x, y) is any point on the line of action of the resultant.

Sum of the moments about an arbitrary point.

Let us find the sum of the moments of the forces about an arbitrary point $A(a, b)$ in terms of X, Y, G.

Their sum is $\sum \overline{AA_r} \times \overline{F_r}$

$$= \sum (\overline{OA_r} - \overline{OA}) \times \overline{F_r}.$$

$$= - \sum \overline{OA} \times \overline{F_r} + \sum \overline{OA_r} \times \overline{F_r}.$$

$$= \sum \left[(-ai - bj) \times (P_r \vec{i} + Q_r \vec{j}) \right] + (\sum G_r) \vec{k}.$$

$$= -a (\sum Q_r) \vec{k} + b (\sum P_r) \vec{k} + \sum G_r \vec{k}.$$

$$= \left[-a (\sum Q_r) + b (\sum P_r) + (\sum G_r) \right] \vec{k}$$

$$= (-aY + bX + G) \vec{k}$$

The sum of the scalar moments about A is

$$-aY + bX + G.$$

$$= \begin{vmatrix} X & Y \\ a & b \end{vmatrix} + G, \quad \text{where } (a, b) \text{ is the arbitrary point.}$$

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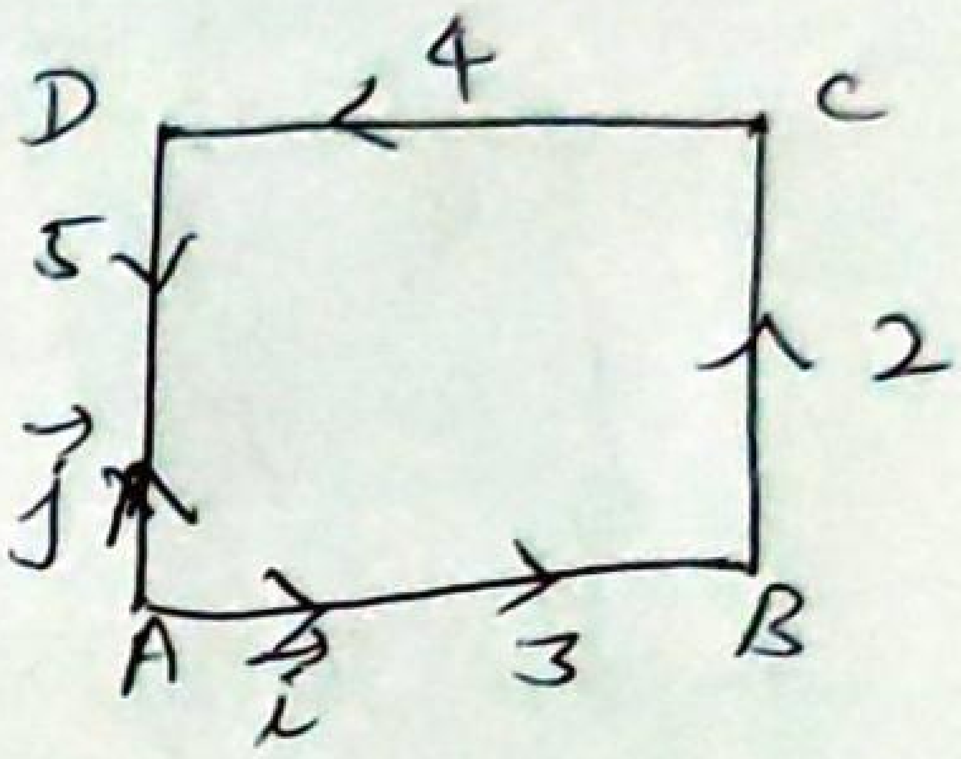
Problem/ D 1

Forces 3, 2, 4, 5 kg. wt. act along the sides AB, BC, CD, DA of a square. Find their resultant and its line of action.

Solution:

Let $\vec{i}$, $\vec{j}$ be the unit vectors parallel to $\overline{AB}$, $\overline{AD}$ and $AB = a$.



Let AB be x axis and AD be y axis.

The vector sum of the forces

$= (3\vec{i}) + (2\vec{j}) + (-4\vec{i}) + (-5\vec{j})$

$= -\vec{i} - 3\vec{j}$.

Let X, Y be the sums of the $\vec{i}$, $\vec{j}$ Components of the forces.

Let G be the sum of the moments about the origin A.

Then $X = -1$, $Y = -3$.

The magnitude of the resultant force

$= \sqrt{X^2+Y^2} = \sqrt{(-1)^2+(-3)^2} = \sqrt{10}$.

$G = 0\times 3 + a(2) + a(4) + 0\times 5$

$= 6a$.

The equation of line of action of the resultant force is $\begin{vmatrix} X & Y \\ x & y \end{vmatrix} + G = 0$.

$\Rightarrow \begin{vmatrix} -1 & -3 \\ x & y \end{vmatrix} + 6a = 0$.

$\Rightarrow \boxed{-y + 3x + 6a = 0}$.

Problem (D2)

If six forces, of relative magnitudes 1, 2, 3, 4, 5 and 6 act along the sides of a regular hexagon, taken in order, show that the single equivalent force is of relative magnitude 6 and that it acts along a line parallel to the force 5 at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of the centre from a side.

Solution:

Let ABCDEF be a regular hexagon.

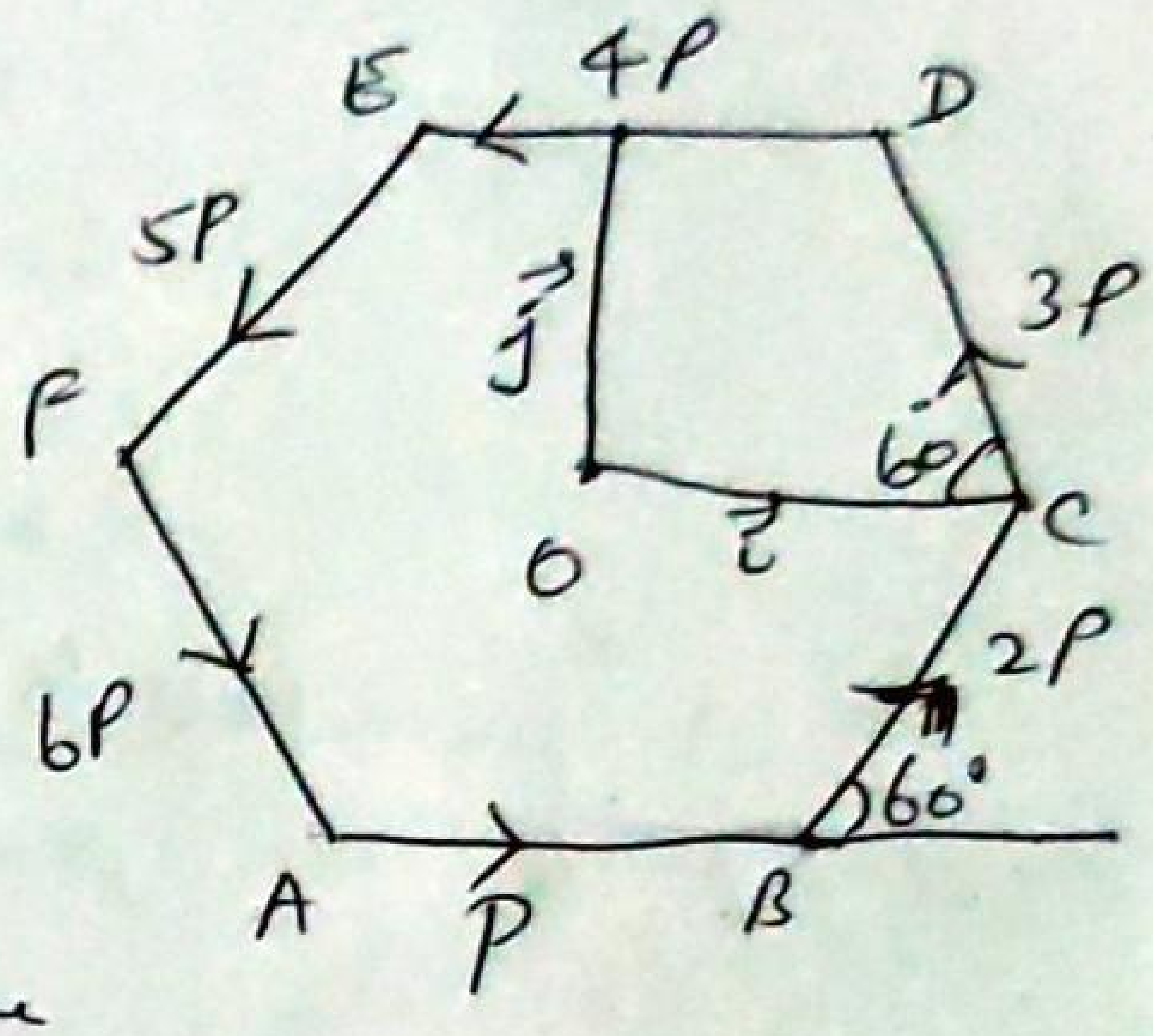
Let O be the centre.

Let the forces be

P, 2P, 3P, 4P, 5P and 6P

along the sides

AB, BC, CD, DE, EF, FA.



Choose OC as x axis, and the perpendicular line through O as y axis.

Let $\vec{i}$, $\vec{j}$ be the unit vectors in their directions.

Now $\hat{AB} = \vec{i}$

$\hat{BC} = \cos 60^\circ \vec{i} + \sin 60^\circ \vec{j} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$

$\hat{CD} = -\cos 60^\circ \vec{i} + \sin 60^\circ \vec{j} = -\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$

$\hat{DE} = -\hat{AB}$

$\hat{EF} = -\hat{BC}$

$\hat{FA} = -\hat{CD}$.

Denote

X = sum of the $\vec{i}$ components of the forces.

Y = sum of the $\vec{j}$ components of the forces.

G = sum of the moments of forces about O.

We have

$X = P + 2P(\frac{1}{2}) + 3P(-\frac{1}{2}) + 4P(-1) + 5P(-\frac{1}{2}) + 6P(\frac{1}{2})$

$= -3P.$

$Y = 2P(\frac{\sqrt{3}}{2}) + 3P(\frac{\sqrt{3}}{2}) + 5P(-\frac{\sqrt{3}}{2}) + 6P(-\frac{\sqrt{3}}{2})$

$= -3\sqrt{3}P.$

Resultant force $= X\vec{i} + Y\vec{j}$

$= -3P\vec{i} - 3\sqrt{3}P\vec{j}.$

$= 6P\left(-\frac{\vec{i}}{2} - \frac{\sqrt{3}}{2}\vec{j}\right)$

$= 6P\,\hat{EF}.$

It is parallel to EF and its magnitude $= 6P.$

Let p be the perpendicular distances of the sides from O.

The sum of the moments G of the six forces about O is

$G = p.P + p.2P + p.3P + p.4P + p.5P + p.6P.$

$= 21\,p.P.$

So the equation of the line is

$Yx - Xy = G$

$\Rightarrow -3\sqrt{3}Px + 3Py = 21Pp$

$\Rightarrow 3\sqrt{3}x + y = 7p.$

$\therefore$ Distance of this line from O $\Big\} = \frac{7p}{\sqrt{(-\sqrt{3})^2+1}}$

$= \frac{7p}{2}.$

Problem (D3)

Forces with components (1,0), (−2,0), (1,1) acts respectively at the points (0,0), (1,1), (1,0). What is the system equivalent to?

Solution:

Let X = sum of $\vec{i}$ components
Y = sum of $\vec{j}$ components
G = sum of moments about origin.

Now the forces, having components (1,0), (−2,0), (1,1) are
$\vec{i}$, $-2\vec{i}$, $\vec{i}+\vec{j}$

$\therefore X = 1-2+1=0$, $Y=1$.

The magnitude of the Resultant force $=\sqrt{X^2+Y^2}$
$=1$.

The position vectors of the points of application of the forces are $\vec{0}$, $\vec{i}+\vec{j}$, $\vec{i}$.

So the sum of their moments about the origin

$$= \bar{0}\times\vec{i} + (\vec{i}+\vec{j})\times(-2\vec{i}) + \vec{i}\times(\vec{i}+\vec{j})$$
$$=\bar{0}+2\bar{k}+\bar{k}$$
$$=3\bar{k}$$

So $G=3$.

Thus the equation of motion of the line of action of the resultants force is

$$\begin{vmatrix} X & Y \\ x & y \end{vmatrix} + G = 0 \Rightarrow \begin{vmatrix} 0 & 1 \\ x & y \end{vmatrix} + 3 = 0$$

$$\Rightarrow \boxed{x=3}$$

Problem (D4):

The sums of moments of a given system of coplanar forces about the three points $(-2,0)$, $(0,3)$, $(2,4)$ are $6, 3, -2$ units. Find the magnitude of the resultant force of the system and the equation to its line of action.

Solution:

Let X = sum of $\vec{i}$ components of the forces.

Y = sum of $\vec{j}$ components of the forces.

G = the sum of the moments of the forces about the origin.

Considering the moments about $(-2,0)$, $(0,3)$, $(2,4)$ namely $6, 3$ and -2, we get

$$6 = \begin{vmatrix} X & Y \\ -2 & 0 \end{vmatrix} + G = 2Y + G \rightarrow (1)$$

$$3 = \begin{vmatrix} X & Y \\ 0 & 3 \end{vmatrix} + G = 3X + G \rightarrow (2)$$

$$-2 = \begin{vmatrix} X & Y \\ 2 & 4 \end{vmatrix} + G = 4X - 2Y + G \rightarrow (3)$$

$(2) - (1) \Rightarrow 3X - 2Y = -3$

$(3) - (2) \Rightarrow X - 2Y = -5$

solving, we get $X = 1$, $Y = 3$, $G = 0$.

So, the magnitude of the resultant $= \sqrt{1^2 + 3^2} = \sqrt{10}$.

the equation of the line of action of the force is

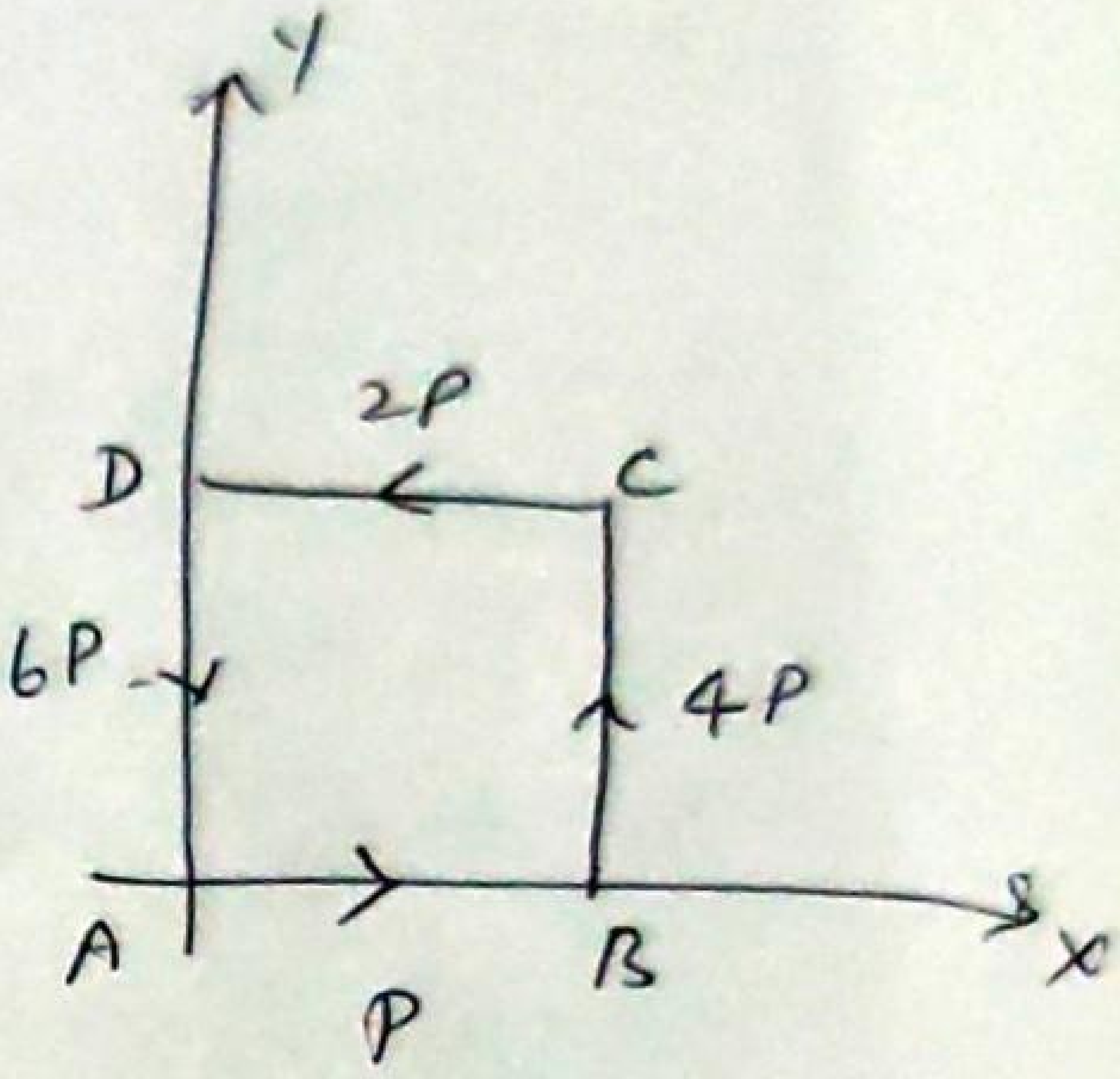
$$\begin{vmatrix} X & Y \\ x & y \end{vmatrix} + G = 0 \Rightarrow \begin{vmatrix} 1 & 3 \\ x & y \end{vmatrix} + 0 = 0$$

$$\Rightarrow \boxed{3x - y = 0}$$

Problem D5:

Forces $P\,\hat{AB}$, $4P\,\hat{BC}$, $2P\,\hat{CD}$, $6P\,\hat{DA}$ act at A, B, C, D of a square of side a. Show that the magnitude of their resultant is $P\sqrt{5}$ and that the equation of its line of action, referred to AB, AD as the x, y axes, is $2x - y + 6a = 0$.

Solution:



Take $AB = X$ axes,
$AD = Y$ axes.

Resolving along AX,

$$X = P - 2P = -P.$$

Resolving along AY,

$$Y = 4P - 6P$$
$$= -2P.$$

G = algebraic sum of moments of the forces about A.

$$= 4Pa + 2Pa = 6Pa.$$

The magnitude of the resultant force at A

is $R = \sqrt{X^2+Y^2} = \sqrt{P^2+4P^2} = \sqrt{5P^2}$

$$= P\sqrt{5}$$

$\therefore$ The system reduces to a force $P\sqrt{5}$ at O, with a couple $6Pa$.

The equation of the line of action of the resultant is $G - x\cancel{Y} + yX = 0$.

$$6Pa - x\cdot(-2P) + y(-P) = 0.$$

$$\Rightarrow 6Pa - x\cdot(-2P) + y(-P) = 0$$

$$\Rightarrow \boxed{2x - y + 6a = 0.}$$

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(E) Equilibrium of a rigid body under three Coplanar forces.

Bookwork : (11)

If three coplanar forces keep a rigid body in equilibrium, then either they all are parallel to one another or they are Concurrent.

Proof: Let the forces be $\overline{F_1}, \overline{F_2}, \overline{F_3}$.

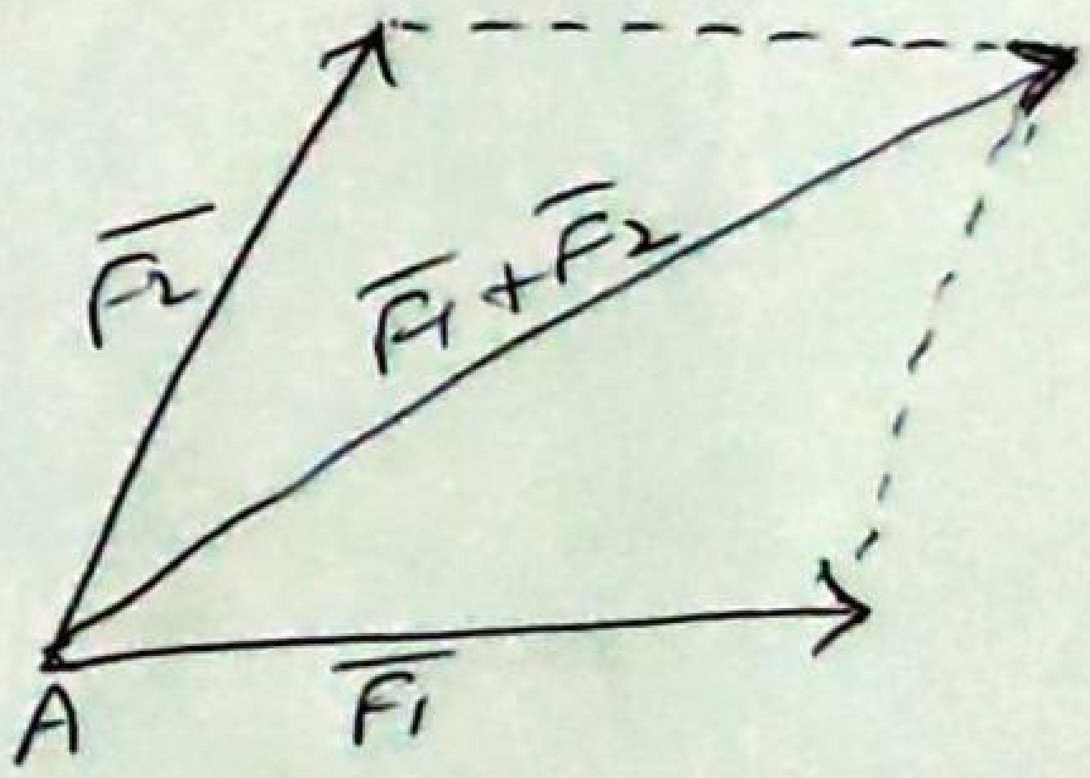
Considering only $\overline{F_1}$ and $\overline{F_2}$, we get the following two cases.

(i) $\overline{F_1}$ and $\overline{F_2}$ are parallel.

(ii) $\overline{F_1}$ and $\overline{F_2}$ are not parallel.

To find: the nature of F_3 in the above two cases.

case (i) $\overline{F_1}$ and $\overline{F_2}$ are parallel



Suppose $\overline{F_1} = F_1\,\vec{i}$

and $\overline{F_2} = F_2\,\vec{i}$ act at A_1 and A_2.

Then their resultant $= (F_1+F_2)\,\vec{i}$

Consequently this resultant $(F_1+F_2)\,\vec{i}$ and $\overline{F_3}$ keep the body in equilibrium.

This implies not only that these two forces act along the same line but also that $\overline{F_3} = -(F_1+F_2)\,\vec{i}$.

So $\overline{F_3}$ is parallel to $\overline{F_1}$ and $\overline{F_2}$.

$\Rightarrow$ The three given forces are parallel to one another.

case (ii)

Suppose the lines of action of $\overline{F_1}$ and $\overline{F_2}$ intersect at A.

Then their resultant $= \overline{F_1} + \overline{F_2}$ acting at A.

This resultant and the third force keep the body in equilibrium.

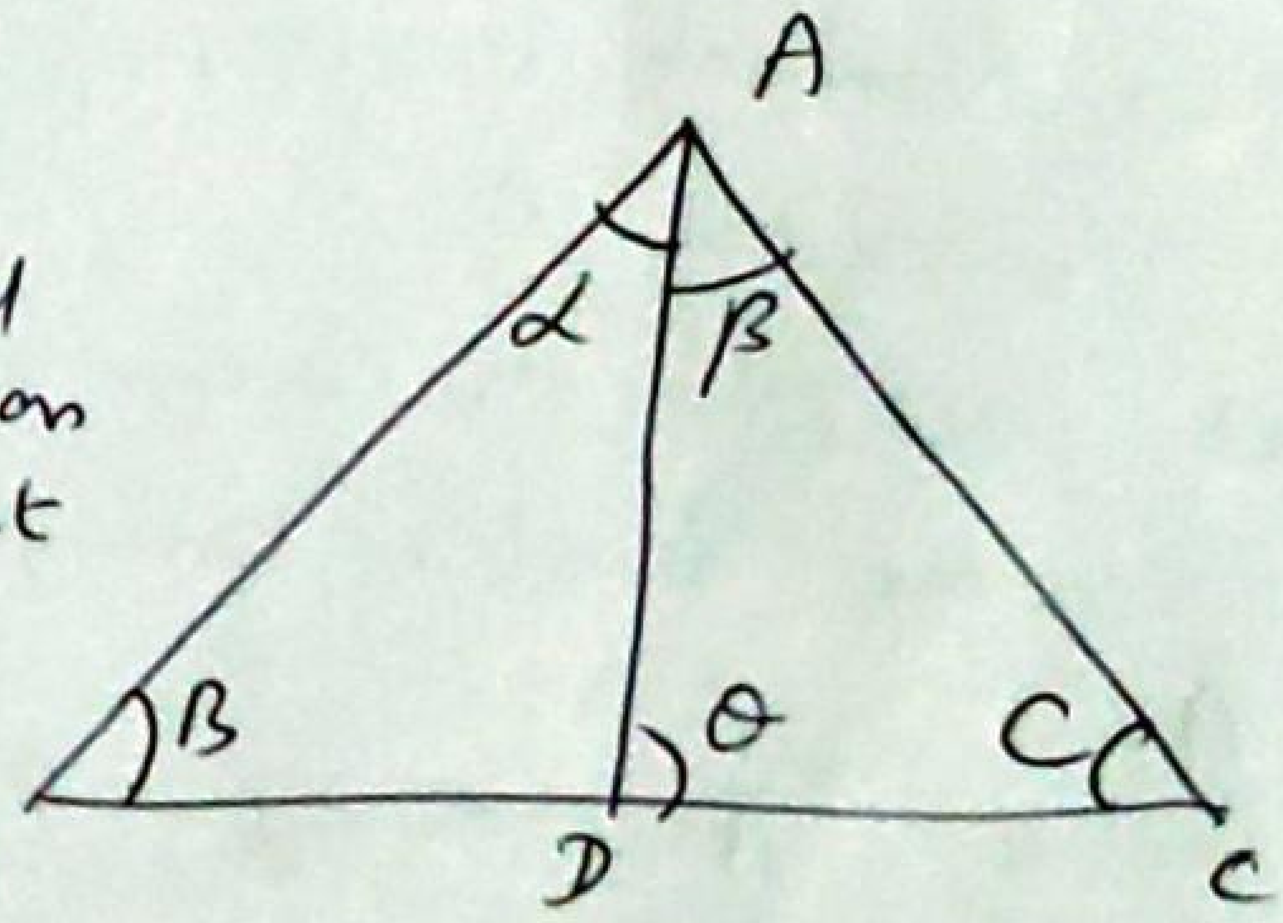
It means that necessarily $\overline{F_1} + \overline{F_2}$ and $\overline{F_3}$ should act along the same line.

$\Rightarrow$ $\overline{F_3}$ also passes through A.

So the forces are concurrent at A.

Thus, we get either the forces are all parallel to one another or they are concurrent.

Cotangent formula



To solve sums when a rigid body is in equilibrium, acted on by three coplanar concurrent forces, (both the cases in which the inclination of the body to the horizontal or vertical is required) we use the trigonometric formula

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta.$$

$$\left(\frac{1}{m}+\frac{1}{n}\right)\cot\theta = \frac{1}{m}\cot B - \frac{1}{n}\cot C,$$

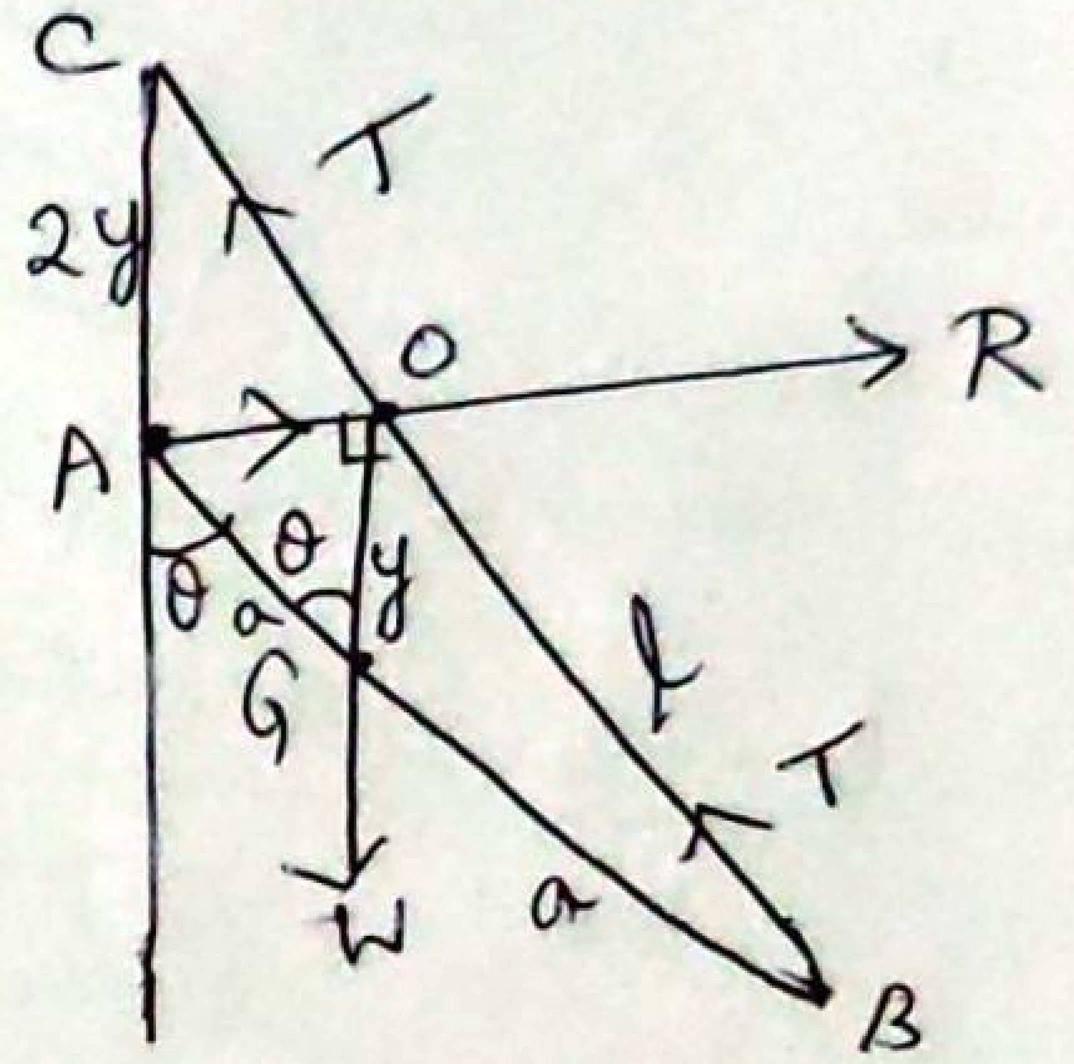
where ABC is a triangle, in which D divides BC internally in the ratio $m:n$ and angle is θ.

Problem (E1):

A uniform rod AB of length 2a hangs against a smooth vertical wall, being supported by a string of length 2l tied to one end of the rod with the other end of the string being attached to a point C in the wall above the rod. Show that the rod can rest inclined to the wall at an angle θ, where $\cos^2\theta = \frac{l^2-a^2}{3a^2}$.

Solution:

From the figure, AB = rod = $2a$
BC = string = $2l$.



the forces acting on the rod are

(i) Weight of the rod at G

(ii) Tension of the string at B.

(iii) Normal reaction of the wall at A.

Let these 3 forces concur at O.

From the similar triangles GBO, ABC,

if $AC = 2y$, $OG = y$.

From the right angled Δ^{le} AGO,

$$\cos\theta = \frac{OG}{AG} = y/a \Rightarrow \cos^2\theta = y^2/a^2 \rightarrow (1)$$

From the right angled triangles AOC, AOG

$$AO^2 = l^2 - (2y)^2 = l^2 - 4y^2 \qquad \rightarrow (2)$$

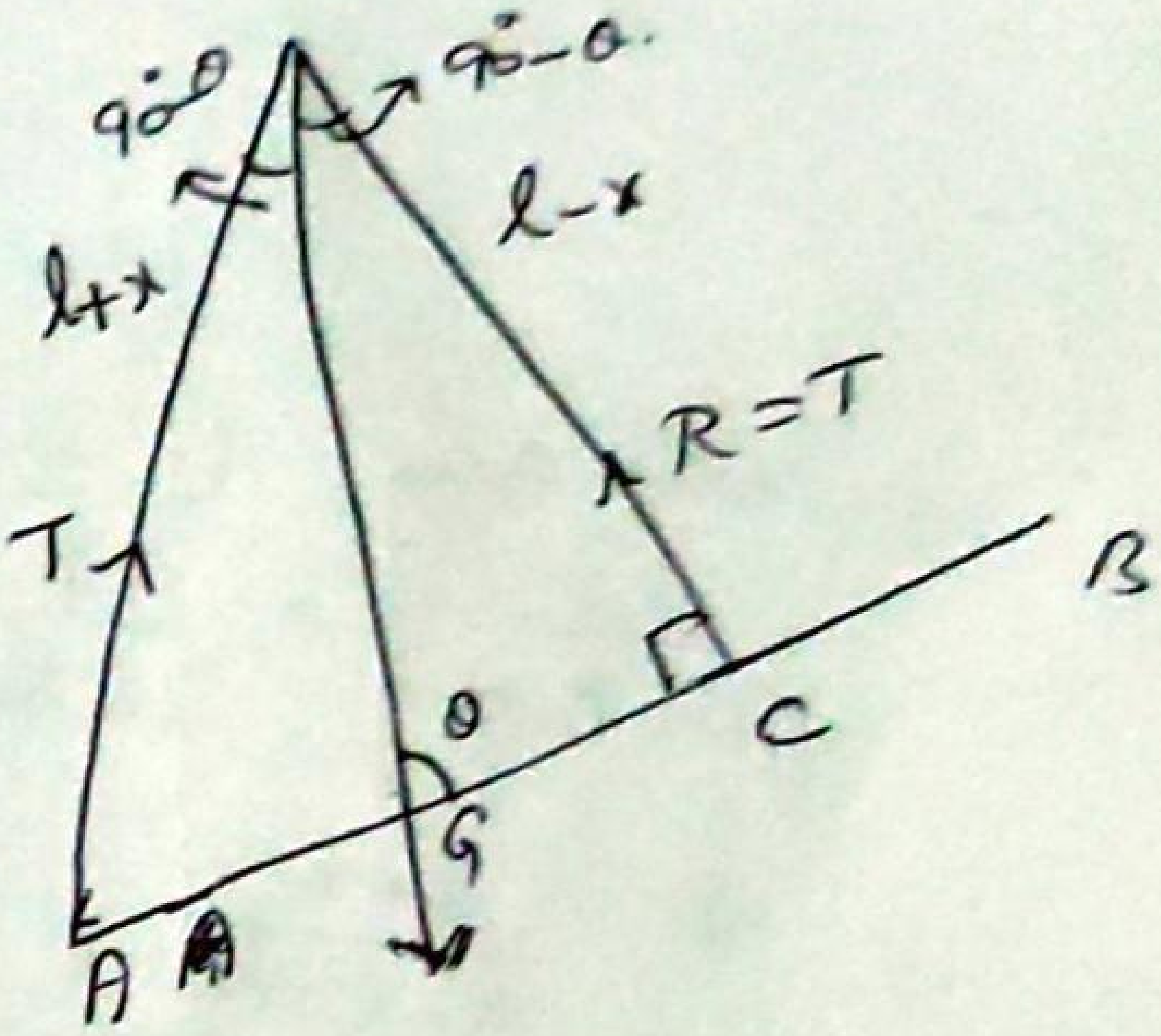
and $$AO^2 = a^2 - y^2. \qquad \rightarrow (3)$$

(2) & (3) $\Rightarrow$ $$y^2 = \frac{l^2-a^2}{3}$$

$$\therefore \boxed{\cos^2\theta = \frac{l^2-a^2}{3a^2}} \text{ by } (1)$$

Problem (E2):

A string of length $2l$ has one end attached to the extremity of a smooth heavy rod AB of length $2a$ and the other end carries a weightless ring C which slides on the rod. The string is hung over a smooth peg O. Show that, if θ is the angle which the rod makes with the vertical, then $l\cos\theta = a\sin^3\theta$.



Solution:

The forces acting on the rod are

(i) Tension T at A along AO.

(ii) Reaction R of the ring along CO and $\perp$r to AB.

(iii) Weight W along OG.

The tension in the string is uniform. So the tension at both the ends ~~of~~ of the string are equal.

The ring is at rest.

The ring at the end $= T$

So $R = T$.

(i.e) the tension of the string at either side of pegs are T. and the tension ~~on~~ bisects $\angle AOC$.

So OG bisects $\angle AOC$.

$\therefore \angle AOG = \angle COG = 90^\circ - \theta$.

Let $AO = l+x$ and $OC = l-x$.

Then from the triangle AOC,

$$\frac{OC}{OA} = \sin(2\theta - 90^\circ).$$

$$\Rightarrow \frac{l-x}{l+x} = -\cos 2\theta.$$

$$\therefore \quad \Rightarrow l - x = -\cos 2\theta (l+x).$$

$\Rightarrow l - x = -l\cos 2\theta + x\cos 2\theta.$

$\Rightarrow -x + x\cos 2\theta = -l\cos 2\theta - l.$

$\Rightarrow x(-1+\cos 2\theta) = -l(\cos 2\theta + 1)$

$\Rightarrow x = -\dfrac{l(1+\cos 2\theta)}{-1+\cos 2\theta}$

$\Rightarrow x = l\dfrac{(1+\cos 2\theta)}{1-\cos 2\theta}$

$= l\dfrac{2\cos^2\theta}{2\sin^2\theta}$

$= l\cot^2\theta. \longrightarrow (1)$

Also, from triangle AOG, using sine formula,

$\dfrac{AO}{\sin AGO} = \dfrac{AG}{\sin AOG}.$

$\Rightarrow \dfrac{l+x}{\sin(180^\circ-\theta)} = \dfrac{a}{\sin(90^\circ-\theta)}$

$\Rightarrow \dfrac{l+x}{\sin\theta} = \dfrac{a}{\cos\theta}.$

$\Rightarrow l+x = \dfrac{a}{\cos\theta}.\sin\theta.$

$= a\tan\theta.$

$\Rightarrow x = a\tan\theta - l. \longrightarrow (2).$

(1) & (2) $\Rightarrow l\cot^2\theta = a\tan\theta - l.$

$\Rightarrow l(\cot^2\theta + 1) = a\tan\theta.$

$l\operatorname{cosec}^2\theta = a\tan\theta.$

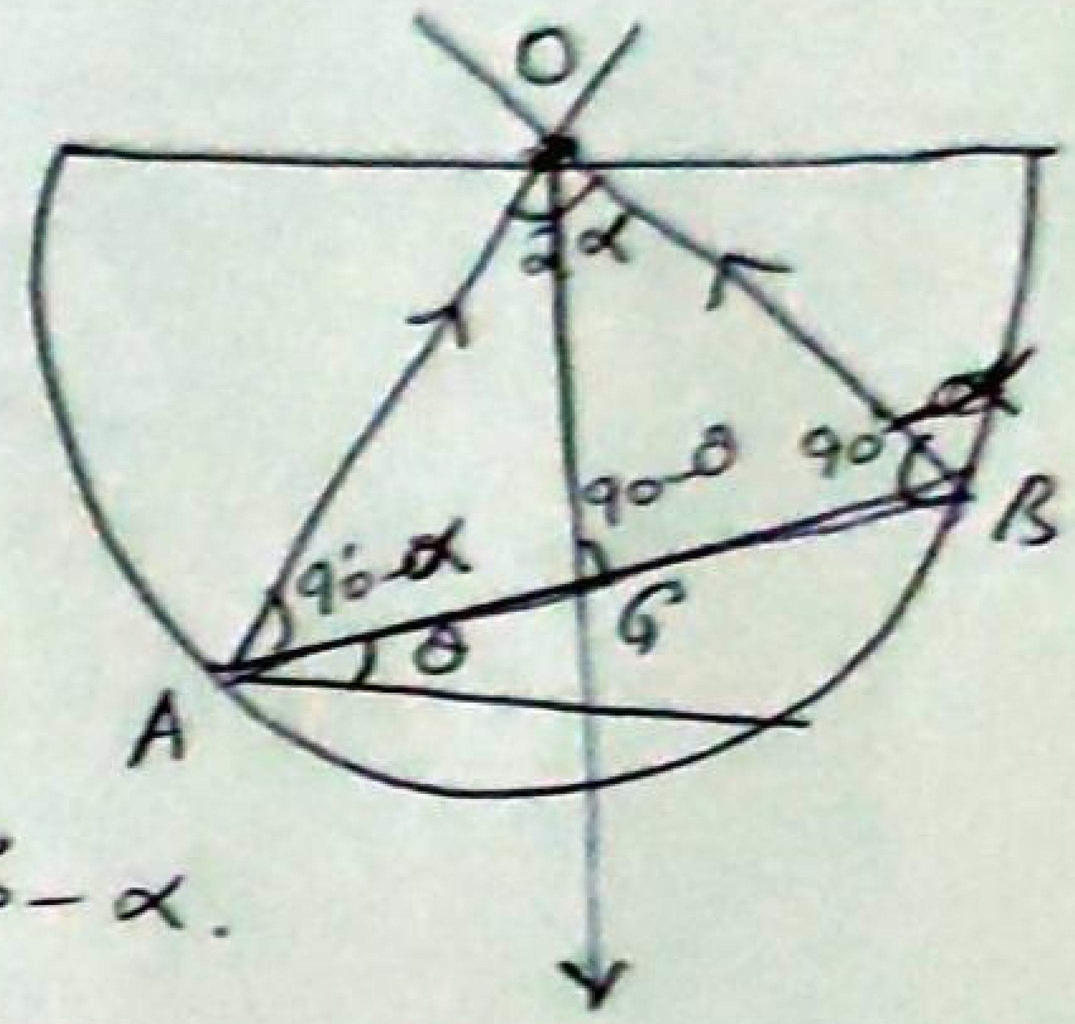
$\Rightarrow \dfrac{l}{\sin^2\theta} = a\dfrac{\sin\theta}{\cos\theta}.$

$\Rightarrow \boxed{l\cos\theta = a\sin^3\theta}$

Problem (E3):

A rod AB rests within a smooth hemispherical bowl. The centre of Gravity G divides it into two portions of lengths a, b. Show that, if 2α is the angle subtended by the rod at the centre of the bowl and θ is the inclination of the rod to the horizon in the equilibrium position, then $\tan\theta = \frac{b-a}{b+a}\tan\alpha$.

Solution:



The forces acting on the rod are

(i) The reaction at A along AO.

(ii) The reaction at B along BO.

(iii) The weight along the vertical OG.

In the triangle OAB, $AG : GB = a : b$

and ΔOAB is isosceles

$\angle OAB = \angle OBA = 90^\circ - \alpha$.

Also $\angle OGB = 90^\circ - \theta$,

By Cotangent formula,

$$\left(\frac{1}{a}+\frac{1}{b}\right)\cot(90^\circ-\theta) = \frac{1}{a}\cot(90^\circ-\alpha) - \frac{1}{b}\cot(90^\circ-\alpha).$$

$$\Rightarrow (a+b)\tan\theta = b\tan\alpha - a\tan\alpha.$$

$$\Rightarrow \boxed{\tan\theta = \frac{b-a}{b+a}\tan\alpha}.$$

Problem (E4):

A uniform beam of weight W hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle α with the horizon. Show that the reaction R at the hinge and tension T on the string are $R = \frac{W}{4}\sqrt{8+\operatorname{cosec}^2\alpha}$, $T = \frac{W}{4}\operatorname{cosec}\alpha$.

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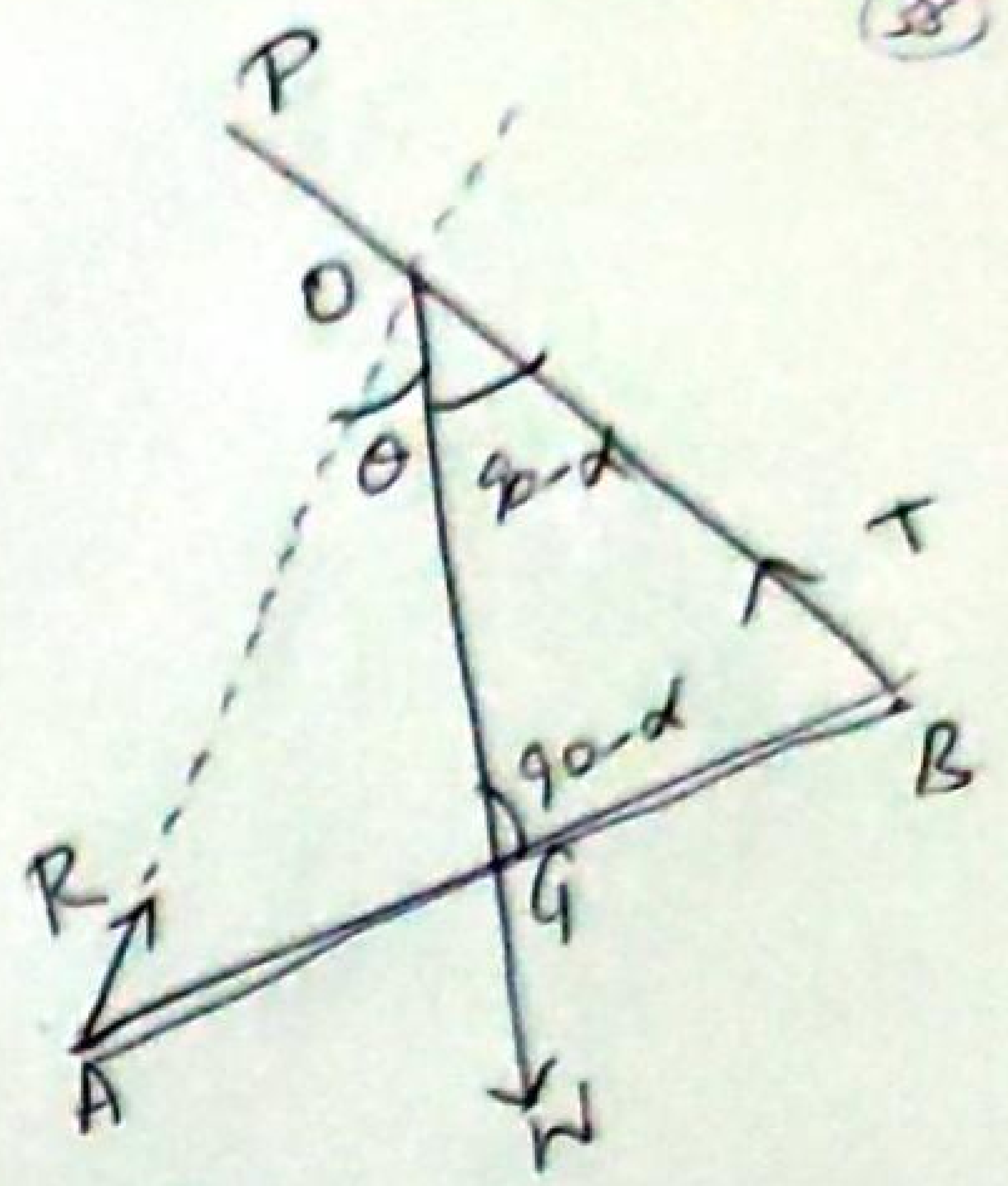
Solution:

Let AB be the beam,

G its centre of mass,

RP = string.



Let the weight and tension meet at O.

Then the reaction of the hinge passes through O.

Since the rod and the string are equally inclined to the horizontal at an angle α, they are equally inclined to the vertical OG at an angle $90^\circ - \alpha$.

Let $\angle AOG = \theta$.

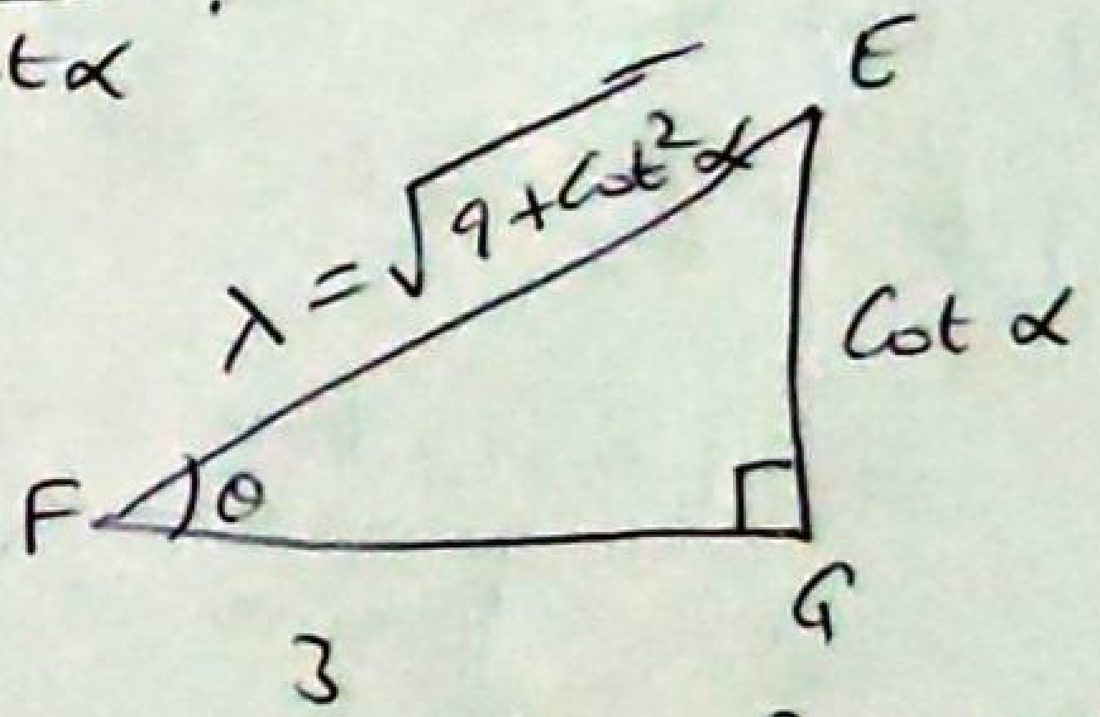
Then by cotangent formula,

$$(1+1)\cot(90^\circ-\alpha) = \cot\theta - \cot(90^\circ-\alpha)$$

$$\Rightarrow 2\cot(90^\circ-\alpha) + \cot(90^\circ-\alpha) = \cot\theta.$$

$$\Rightarrow 3\tan\alpha = \cot\theta$$

$$\Rightarrow \cot\theta = \frac{3}{\cot\alpha}.$$



Take $\lambda = \sqrt{9+\cot^2\alpha}$.

$\therefore \sin\theta = \dfrac{\cot\alpha}{\lambda}$, $\cos\theta = \dfrac{3}{\lambda}$.

$FG^2 + GE^2 = FE^2$.

Now the angles opposite to

$R = 90^\circ + \alpha$

$T = 180^\circ - \theta$

$W = 90^\circ + \theta - \alpha$.

Then by Lami's theorem,

$$\frac{R}{\sin(90^\circ+\alpha)} = \frac{T}{\sin(180-\theta)} = \frac{W}{\sin(90^\circ+\theta-\alpha)}.$$

$$\Rightarrow \frac{R}{\cos\alpha} = \frac{T}{\sin\theta} = \frac{W}{\cos(\theta-\alpha)} \rightarrow (1)$$

In this, Considering $\cos(\theta-\alpha)$ separately,

$$\cos(\theta-\alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha.$$
$$= \frac{3}{\lambda}\cos\alpha + \frac{\cot\alpha}{\lambda}\sin\alpha$$
$$= \frac{4\cos\alpha}{\lambda}$$

$\therefore$ from the I & III of ①,

$$R = \frac{W\cos\alpha}{\left(\frac{4\cos\alpha}{\lambda}\right)} = \frac{W\lambda}{4}$$
$$= \frac{W}{4}\sqrt{9+\cot^2\alpha}.$$
$$= \frac{W}{4}\sqrt{8+(1+\cot^2\alpha)}$$
$$\boxed{R = \frac{W}{4}\sqrt{8+\operatorname{cosec}^2\alpha}.}$$

$$T = \frac{W\sin\theta}{\left(\frac{4\cos\alpha}{\lambda}\right)} = \frac{W}{4}\cdot\frac{\lambda}{\cos\alpha}\cdot\frac{\cot\alpha}{\lambda}.$$
$$\boxed{T = \frac{W}{4}\operatorname{cosec}\alpha}$$

Problem (E5):

A uniform rod AB rests with a point P on it $(AP = \frac{3}{4}AB)$ in contact with a fixed smooth peg and the end A attached to a light string which is fastened to a fixed point. If the rod makes an angle of 45° with the horizontal, show that the string makes with the horizontal angle whose tangent is 2.

Solution:

Let AB be the rod

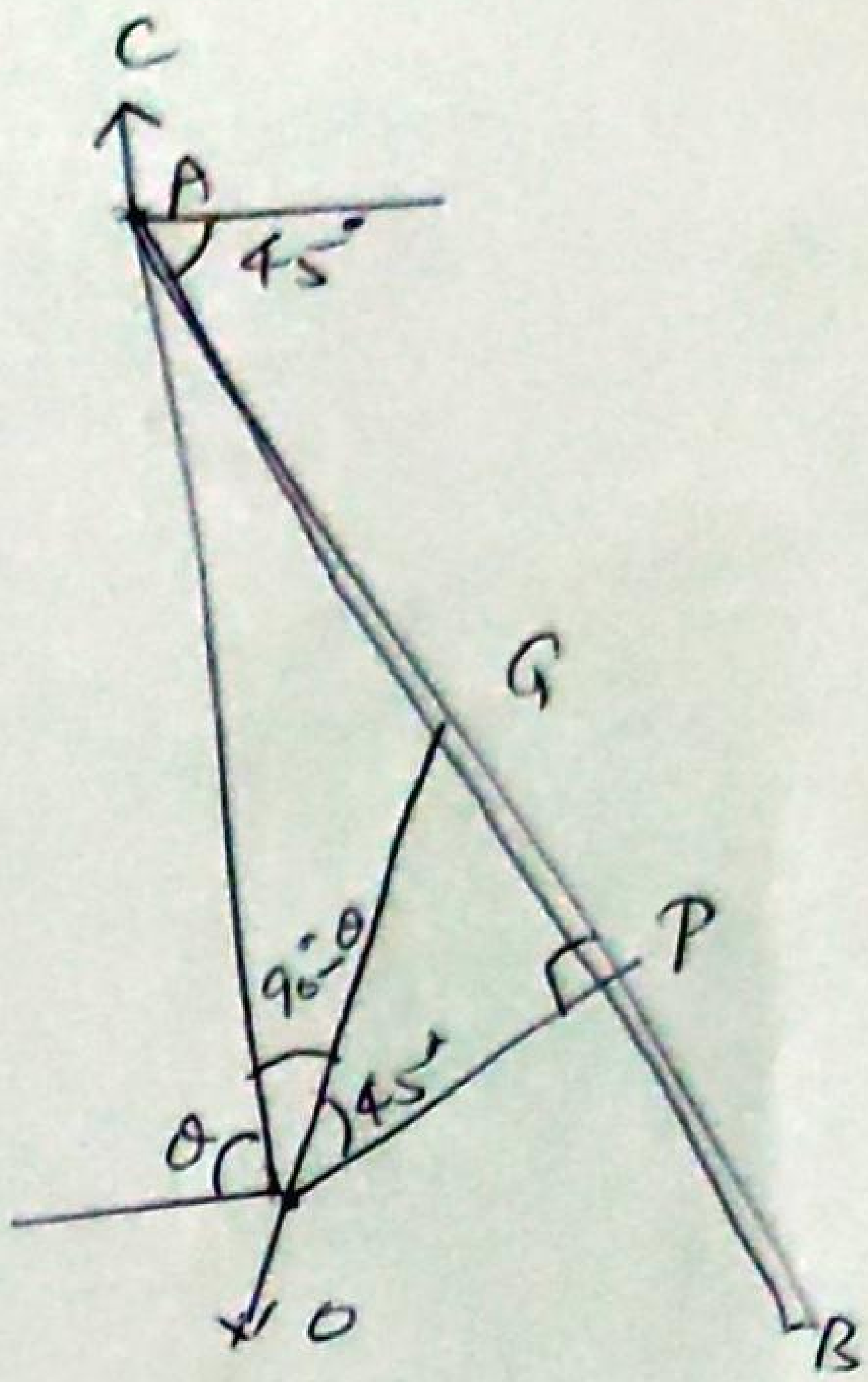
AC the string

G → centre of mass of the rod.

Let the weight of the rod and the normal reaction ~~meet~~ of the peg meet at O.

Then the tension acts along OAC.

Let θ be the ~~angle~~ angle made by the string with the horizontal



Now $\angle AOG = 90° - \theta$.

$\angle GOP = 45°$.

So ΔGOP is isosceles.

We find OP in 2 ways.

(i) $OP = GP = l/4 \quad (AB = l) \longrightarrow (1)$

(ii) $OP = AP \tan(\theta - 45°)$, since $\angle PAO = \theta - 45°$

$= \frac{3l}{4} \frac{\tan\theta - 1}{1 + \tan\theta} \longrightarrow (2) \quad \left[\because \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \right]$

(1) & (2) $\Rightarrow \frac{3l}{4} \frac{\tan\theta - 1}{1 + \tan\theta} = l/4$.

$\Rightarrow 3\tan\theta - 3 = 1 + \tan\theta$.

$\Rightarrow 2\tan\theta = 4$

$\Rightarrow \boxed{\tan\theta = 2}$

~ x ~